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APPROXIMATION PROPERTIES OF THE h-p VERSION OF THE FINITE ELEMENT METHOD

by

I. Babuška

and

B. Q. Guo

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Approximation properties of the h-p version of the finite element method

I. Babuška[1]
Institute of Physical Science and Technology
and Department of Mathematics
University of Maryland
College Park, MD 20742, USA.

B.Q. Guo[2]
Department of Applied Mathematics
University of Manitoba Winnipig MB
R3T 2N2 CANADA

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Abstract

The paper surveys the approximation and convergence properties of the h-p version of the finite element method, the basic theoretical results together with the main ideas of the proofs. Numerical examples are given.

1. Introduction

The h-p version of the finite element method is characterized by the family $\mathcal{F} = \{\mathfrak{M}\}$ of meshes $\mathfrak{M}$ covering the domain Ω defining the elements $\mathcal{T}$, $\mathcal{M} = \{\mathcal{T}\}$ and by the distribution $q(\mathfrak{M})$ of the degrees $p(\mathcal{T})$ of the elements $\mathcal{T} \in \mathfrak{M}$. The shape functions are the usual "pull-back" polynomials i.e. mapped polynomials of degree $p(\mathcal{T})$ defined on the standard element T. The degree $p(\mathcal{T})$ can be different in different directions in T. The standard element can be a triangle or square in two dimensions and cube, wedge or simplex (tetrahedron) in the three dimensions. We denote by $S(\mathfrak{M},q) \subset H^1(\Omega)$ the finite element space and by $N(\mathfrak{M},q)$ its dimension.

The main problem in the finite element method is to estimate

$$\inf_{\substack{\chi\in S(\mathfrak{M},q)\\ N(\mathfrak{M},q)\le N\\ \mathfrak{M}\in\mathcal{F}}} |u-\chi|_{H^1(\Omega)} = \Phi(u,\mathcal{F},q,N). \tag{1.1}$$

Assume for example that $\mathcal{F}$ is a (reasonable) family of quasiuniform meshes on a (reasonable) domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$ and that the degree of all elements is the same i.e. $p(\mathcal{T}) = p \le p_0$, $p \ge 1$, $\mathcal{T} \in \mathfrak{M}$. Further assume that $u \in H^s(\Omega)$. Then we have

$$\Phi(u,\mathcal{F},q,N) \le CN^{\mu/n} \|u\|_{H^s} \tag{1.2}$$

with $\mu = \min(p_0, s-1)$.

If we know apriori more about u, for example that u is analytic on $\bar{\Omega}$ then we have

$$\Phi(u,\mathcal{F},q,N) \le CN^{-p_0/n} \tag{1.3}$$

where C is independent of N.

Assume now that we will consider the family of quasiuniform meshes as before with the uniform degrees of the elements but assume only that $p \ge 1$. Then we

have for u analytic on $\bar{\Omega}$

$$(1.4) \qquad \Phi(u,\mathcal{F},q,N) \le C\, e^{-\gamma N^{1/n}}$$

where C and γ are independent of N. (The mesh achieving (1.4) is a fixed one and $p(\mathcal{T}) \longrightarrow 0$ as $N \longrightarrow \infty$.)

We see that the rate of convergence depends on

a) properties of the exact solution u

b) the admissible set of spaces $S(\mathfrak{M},q)$, $\mathfrak{M} \in \mathcal{F}$.

We do not know the exact solution in advance, but for a large class of engineering problems we can find a characterization of the solution which can be exploited. Obviously the assumption that $u \in H^s(\Omega)$ or u is analytic on $\bar{\Omega}$ is an example of a characterization of solutions of interest.

In practice we have to consider a family of problems characterized by the useful data. Typically in the structural mechanics we deal with the problem characterized by piecewise analytic data, i.e. the boundary of the domain is piecewise analytic as well the material properties, loads, etc. For this class of problems the assumption that $u \in H^s(\Omega)$ or u is analytic on $\bar{\Omega}$ is very inappropriate. If the domain Ω has corners in two dimensions or vertices and edges in 3 dimensions then u is not analytic on $\bar{\Omega}$ and $u \in H^s$ only for small s and the fact that the solution u is analytic inside Ω is not employed.

In this paper we will survey the major results concerning the performance of the h-p version for solving elliptic problems with piecewise analytic input data. We will show that for all these problems the h-p version leads to exponential convergence with respect to the number of degrees of freedom. More precisely we will show that for a family $\mathcal{F}$ of meshes and degrees distribution $q(\mathfrak{M})$ we have

$$(1.5) \qquad |u - u_{FE}|_{H^1} \le \bar{C}\,\Phi(u,\mathcal{F},q,N) \le C\,e^{-\gamma N^{\alpha}}$$

where $\alpha = 1/3$ in two dimensions and $\alpha = 1/5$ in three dimensional problems. Constants C and $\gamma > 0$ are independent of N. They depend on the solution, the domain Ω, the distortion of the used elements, the family $\mathcal{F}$ of used meshes as well the distribution $q(\mathfrak{M})$ of the degree of elements. The family of meshes which lead to (1.5) is a special one and the mesh includes in 3 dimensions the "needle" elements along the edges of the domain.

We will discuss separately the two and three dimensional case and present the major ideas and results, but without detailed proofs which will be referred to. The two dimensional case will be discussed in more details because it offers some analogies for the 3 dimensional case. The numerical examples in 2 and 3 dimensions will be given also.

The survey papers about various aspects of the h,p and h-p version we refer to [1] [2] [3].

2. The h-p FE version in two dimensions

2.1. Preliminaries.

Let $\Omega \subset \mathbb{R}^2 = \{x_1, x_2\} = \{x \mid x_i \in \mathbb{R},\ i = 1,2\}$ be a bounded domain with the boundary $\partial\Omega$. We will assume that $\partial\Omega = \bigcup_{i=1}^{R} \Gamma^{(i)} = \Gamma$ with $\Gamma^{(i)}$ being Jordan curves and

$$\Gamma^{(i)} = \bigcup_{j=1}^{M(i)} \bar{\Gamma}_j^{(i)}, \qquad i = 1, \ldots, N$$

where $\bar{\Gamma}_j^{(i)}$ are *analytic arcs* i.e. $\bar{\Gamma}_j^{(i)} = \{x_1 = \varphi_{i,j}(\xi),\ x_2 = \psi_{i,j}(\xi) \mid \xi \in \bar{I} = [-1, 1]\}$ with $\varphi_{i,j}(\xi)$, $\psi_{i,j}(\xi)$ analytic functions on $\bar{I}$ and $|\frac{d}{d\xi} \varphi_{i,j}(\xi)|^2 + |\frac{d}{d\xi} \psi_{i,j}(\xi)|^2 \geq \alpha > 0\}$. Here $(\varphi_{i,j}, \psi_{i,j})$ are Cartesian coordinates of $\Gamma_j^{(i)}$. By $\Gamma_j^{(i)}$ we denote the open arc, i.e. the image of $I = (-1,1)$. The domain Ω is R-connected. We will assume that the curve Γ^1 is the boundary of the infinite component of the complement of Ω.

An example of a domain Ω of interest is given in Fig. 2.1.1. Orientation of the curves is shown in the Figure 2.1.1 also. The endpoints of the arc $\Gamma_j^{(i)}$, $i = 1, \ldots, R$, $j = 1, \ldots, M(i)$ are denoted by $A_{j-1}^{(i)}$, $A_j^{(i)}$ (i.e. $A_{j-1}^{(i)} = (\varphi_{i,j}(-1), \psi_{i,j}(-1))$, $A_j^{(i)} = (\varphi_{i,j}(1), \psi_{i,j}(1))$, $A_0^{(i)} = A_{M(i)}^{(i)}$, and called vertices of Ω. The measure of the internal angle at $A_j^{(i)}$ is denoted by $\omega_j^{(i)}$. We will assume that $0 < \omega_j^{(i)} \leq 2\pi$.

We will also understand the Jordan curve in an obviously generalized sense when parts of different arcs could coincide so that a slit domain could also be considered (with $\omega_j^{(i)} = 2\pi$)

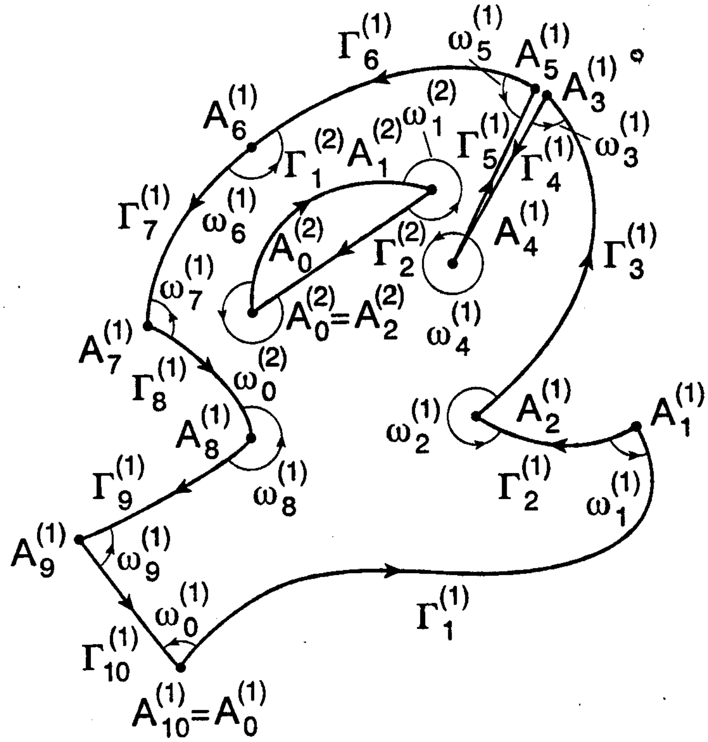

Fig. 2.1.1. The scheme of a domain with a piecewise analytic boundary

Note that we can have $\omega_i^{(j)} = \pi$ at some vertex $A_j^{(i)}$ (for example $A_6^{(1)}$). The vertex has to be introduced when $\Gamma_{j+1}^{(i)}$ is not the analytic continuation of $\Gamma_j^{(i)}$. We can of course place an additional vertex at any place of $\Gamma_j^{(i)}$. This has to be done because below we will assume that boundary condition is analytic and is of the same type on every arc.

Further let $\Gamma = {}^{D}\bar{\Gamma} \cup {}^{N}\Gamma$, ${}^{D}\bar{\Gamma} = \bigcup_{i,j\in\mathcal{D}} \bar{\Gamma}_j^{(i)}$, ${}^{N}\Gamma = \Gamma - {}^{D}\bar{\Gamma} = \bigcup_{i,j\in N}\Gamma_j^{(i)}$ where $\mathcal{D},N$ are the subsets of the set $\{i,j \mid i = 1,\ldots,R,\ j = 1,\ldots,M(i)\}$. For simplicity we will assume that ${}^{D}\Gamma \neq 0$. Otherwise we have to add the usual conditions of solvability and uniqueness.

In Figure 2.1.1 we have $R = 2$. We will use $R = 1$ in what it follows and will write M instead of $M(i)$, A_j instead of $A_j^{(i)}$, etc. We note that

all our arguments and results are valid in the general case too.

If the arcs are the *straight lines* then we say that Ω is a *polygon*. Otherwise we speak about a *curvilinear polygon*.

Let further $B_j^{r_j}$ be the balls with the radius r_j and the center in the vertex A_j and let $\Omega_j^{(r_j)} = B_j^{r_j} \cap \Omega$. We will assume that $r_j < \rho$, $j = 1,\dots,M$, and ρ is sufficiently small so that $\partial\bar{\Omega}_j^{(r_j)} \cap \Gamma \subset \bar{\Gamma}_{j+1} \cup \bar{\Gamma}_j$, $\bar{\Omega}_j^{(r_j)} \cap \bar{\Omega}_i^{(r_i)} = \emptyset$ for every $i \neq j$ and where $\partial\bar{\Omega}_j^{(r_j)}$ is the boundary of $\Omega_j^{(r_j)}$. Finally we denote $\Omega_0^{(r)} = \Omega - \bigcup_{j=1}^{M} \bar{\Omega}_j^{(r_j)}$ with $r = (r_1,\dots,r_M)$.

The typical example is shown in the Figure 2.1.2.

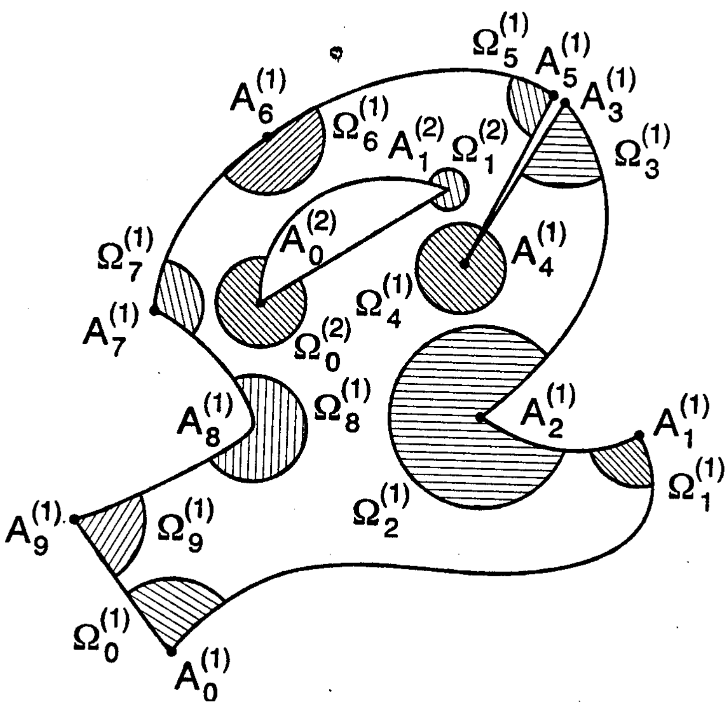

Figure 2.1.2. The partition of the domain Ω.

By $H^1(\Omega)$ we denote the usual Sobolev space and $H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u=0 \text{ on } {}^D\Gamma\}$. Let us now define the space $\mathfrak{L}(\beta) \subset H_D^1(\Omega)$, $\beta = (\beta_1,\dots,\beta_M)$, $0 < \beta_i < 1$, $i = 1,\dots,M$. For $\alpha = (\alpha_1,\alpha_2)$, $\alpha_i \geq 0$ integers, $i = 1,2$, $|\alpha| = \alpha_1 + \alpha_2$ let

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}_{x_1}\,\partial^{\alpha_2}_{x_2}}$$

and for $m \ge 0$ an integer let

$$|D^m u|^2(x) = \sum_{|\alpha| = m} |D^{\alpha}u|^2(x).$$

Further denote

$$\Phi_j(x) = |x - A_j| = \text{dist } (x,A_j),\ j = 1,\dots,M.$$

Let u be such that there exist constants C_j, d_j, and $0 < \beta_j < 1$, $j = 1,\dots,M$, $j = 0,\dots,M$ so that for any $\alpha = (\alpha_1,\alpha_2)$, $\alpha_i \ge 0$ integers $i = 1,2$:

i) for $x \in \Omega_j^{(r_j)}$:

(1.1) $$|D^{\alpha}u|(x) \le C_j d_j^{\alpha}\,\alpha!\ \Phi_j^{-(\beta_j+\alpha_1+\alpha_2-1)}(x)$$

ii) for $x \in \Omega_0^{(r/2)}$

(1.2) $$|D^{\alpha}u|(x) \le C_0 d_0^{\alpha}\,\alpha!$$

where $d_j = (d_j^{(1)}, d_j^{(2)})$, $d_j^{(i)} > 1$, $d^{\alpha} = (d_j^{(1)})^{\alpha_1}(d_j^{(2)})^{\alpha_2}$, $\alpha! := \alpha_1!\alpha_2!$, $0! = 1$. By $\mathfrak{K}(\beta)$ we denote the set of all functions u satisfying (1.1), (1.2).

Let us underline that $\Omega_0^{(r/2)}$ overlaps the domains $\Omega_j^{(r_j)}$. Hence in this overlap area the function $u \in \mathfrak{K}(\beta)$ under consideration satisfies both conditions i) and ii). We always will assume an overlap without mentioning it explicitly and write often $\Omega_0^{(r)}$ only. The description of the space $\mathfrak{K}(\beta)$ depends on r, nevertheless this is not essential because the constants C and d in (1.1) and (1.2) are not explicitly specified. On the other hand the dependence on β is essential. The assumption $0 < \beta_i < 1$ guarantees that $\mathfrak{K}(\beta) \subset H^1(\Omega)$ and $\mathfrak{K}(\beta) \subset C^0(\bar{\Omega})$.

2.2. The boundary value problem.

Let us consider the problem

(2.1a) $$-\Delta u = f \quad \text{in } \Omega,$$

(2.1b) $$u = 0 \quad \text{on } {}^D\Gamma,$$

(2.1c) $$\frac{\partial u}{\partial n} = g \quad \text{on } {}^N\Gamma.$$

We understand the problem (2.1) in the weak sense. Find $u \in H_0^1(\Omega)$ such that for any $v \in H_0^1(\Omega)$

$$B(u,v) = \int_\Omega \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} =$$

$$= \int_\Omega f\, v + \int_{{}^N\Gamma} g\, v$$

holds.

We will assume that the functions f and g are such that the solution $u \in \mathfrak{L}(\beta)$ for some β dependent on Ω, more precisely β_i depends only on the internal angle ω_i.

In [4] [5] [6] we have analyzed in detail the space of input data g and f for which $u \in \mathfrak{L}(\beta)$ and gave a detailed description of these spaces. As special case we have shown that $u \in \mathfrak{L}(\beta)$ with properly selected β_i dependent on ω_i if

a) function f is analytic on $\overline{\Omega}$

b) g is analytic on every arc $\overline{\Gamma}_j \subset {}^N\overline{\Gamma}$

Remark 2.1. In [4] we have analyzed the regularity of the solution u of the boundary value problem for general elliptic operator with analytic coefficients on $\overline{\Omega}$ and have described the spaces of g and f that $u \in \mathfrak{L}(\beta)$.

Remark 2.2. In [6] we have analyzed the regularity of the solution u of the elasticity problem and have shown that $u \in \mathcal{B}(\beta)$ when input data are piecewise analytic.

Remark 2.3. In [7] we analyzed the eigenvalue problem. We have shown that the eigenfunctions belong to the space $\mathcal{B}(\beta)$.

Remark 2.4. We restricted ourselves to the case that $u = 0$ on $^{D}\Gamma$. In the above mentioned papers we have shown (as special case) that if $u = \tilde{g}$ with $\tilde{g}$ analytic on the arcs $\Gamma_j \in {}^{D}\Gamma$ we get also $u \in \mathcal{B}(\beta)$. The restriction $u = 0$ on $^{D}\Gamma$ will simplify our approximation theorem. In general when $u \neq 0$ on $^{D}\Gamma$ we have to replace the nonhomogeneous Dirichlet boundary condition by a properly selected sequence of functions. We will not address here this problem. (See for more e.g. in [8]).

Let us underline that in the engineering practice we essentially always deal with piecewise analytical data and hence the solution under consideration always belongs to the space $\mathcal{B}(\beta)$.

It is essential to deal with the spaces of solutions which are as small as possible but will include practically all the cases of engineering importance.

2.3. The elements

As usually, we will assume that the domain Ω is covered by the mesh $\mathfrak{M}$ which partitions the domain Ω into elements $\mathcal{T}$ and will consider a family $\mathcal{F}$ of admissible meshes $\mathfrak{M} \in \mathcal{F}$. Hence we will write $\mathfrak{M} = \{\mathcal{T}\}$. We will consider first family $\mathcal{F}$ of meshes with quadrilateral elements.

Let us denote by D the standard quadrilateral element, $D = I^2$, $I = (-1,1)$ i.e.

$$D = \{\eta_1, \eta_2 \mid |\eta_1| < 1, |\eta_2| < 1\}.$$

We will assume that any $\mathcal{T} \in \mathfrak{M}$ is an image of D by the one to one mapping $M_{\mathcal{T}}$, i.e.

$$\mathcal{T} = \{x_1, x_2 \mid x_1 = X_1(\eta_1,\eta_2), x_2 = X_2(\eta_1,\eta_2), \eta_1, \eta_2 \in D\}.$$

About X_i, $i = 1,2$ we will assume that there is $\nu(\mathcal{T}) > 0$ such that

i) for any $|\alpha| = m > 0$ and any $\eta \in \bar{D}$, $\ell = 1,2$

$$|D^{\alpha}X_{\ell}(\eta)| \le C_0 d_0^m \, m! \, \nu, \quad m = 1,2\ldots \tag{3.1}$$

ii) The Jacobian determinant $\mathcal{T}$ satisfies

$$C_1\nu^2 \le |\mathcal{T}| = \left|\frac{\partial(X_1,X_2)}{\partial(\eta_1,\eta_2)}\right| \le \bar{C}_1\nu^2. \tag{3.2}$$

In (3.1) and (3.2) ν dependents on the element $\mathcal{T}$ but the constants $C_0, C_1, \bar{C}_1, d_0$ are indepndent of $\mathcal{T} \in \mathfrak{M} \in \mathcal{F}$. Obviously υ is related to the size of the element $\mathcal{T}$.

Remark 3.1. If the boundary of the element lies on the boundary Γ then there is obviously a relation between the constants in (3.1) (3.2) and description of the analytic arc Γ_j.

Remark 3.2. We will assume that every vertex of Ω is also the vertex of an element.

Remark 3.3. Let us note that from (3.1) we see that $X_i(\eta), i = 1,2$ are analytic functions on $\bar{D}$ and hence they can be extended beyond $\bar{D}$ in some of neighborhood of $\bar{D}$.

Remark 3.4. Our assumptions imposed on the elements guarantee that the aspect ratio of $\mathcal{T} \in \mathfrak{M} \in \mathcal{F}$ is bounded.

Obviously any quadrilateral with the straight sides satisfies the conditions mentioned above. For example for a_i, b_i, c_i, $i = 1,2$ of order one and

$$x_i = X_i(\eta_1, \eta_2) = x_i^0 + \nu(a_i(1+\eta_1)(1-\eta_2) +$$
$$+ b_i(1+\eta_1)(1+\eta_2) + c_i(1-\eta_1)(1+\eta_2))$$

maps D onto $\mathcal{T}$ (shown in the Figure 2.3.1) provided some conditions on the coefficients a_i, b_i, c_i are imposed.

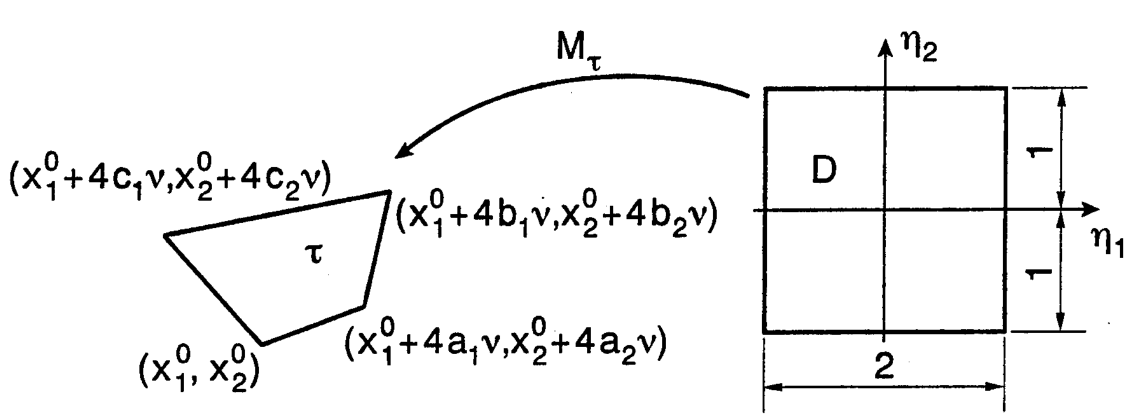

Fig. 2.3.1. The element $\mathcal{T}$

Let now $\mathcal{T}$ be a triangular element. Then we will assume that $\mathcal{T}$ is the image of the master element $T = \{\eta_1, \eta_2 | (\eta_1, \eta_2) \in D,\ \eta_2 < \eta_1\} \subset D$ by the mapping $M_{\mathcal{T}}$ which satisfies (3.1) (3.2) and $M_{\mathcal{T}}(D) \subset \Omega$. Hence $\mathcal{T}$ is a half of $M_{\mathcal{T}}(D)$.

We now describe the elements of the meshes we will consider.

1) Elements in $\Omega_j^{(r_j)}$.

We will consider two kinds of elements $\mathcal{T}$ and $\mathcal{T}_0$ separately.

The element $\mathcal{T}$. Let $\mathcal{T} \subset \Omega_j^{(r_j)}$, then $A_j \notin \overline{\mathcal{T}}$. Denote

(3.3a) $$\underline{\kappa}_j(\mathcal{T}) = \min_{x \in \mathcal{T}} \Phi_j(x),$$

(3.3b) $$\kappa_j(\mathcal{T}) = \max_{x\in\mathcal{T}} \Phi_j(x).$$

We assume that there is a constant A, $1 < A < \infty$ such that

(3.4) $$\frac{\kappa_j(\mathcal{T})}{\underline{\kappa}_j(\mathcal{T})} \le A$$

and

(3.5) $$\underline{C}\ \kappa_j(\mathcal{T}) \le \nu(\mathcal{T}) \le \overline{C}\ \kappa_j(\mathcal{T}).$$

It is easy to see from (3.1) and (3.5) that there exists $A^* > 1$ such that

(3.6) $$A^* \le \frac{\kappa_j(\mathcal{T})}{\underline{\kappa}_j(\mathcal{T})}$$

and with $\|\mathcal{T}\|$ being the measure of $\mathcal{T}$ we get

(3.7) $$\underline{C}\ \kappa_j^2(\mathcal{T}) \le \|\mathcal{T}\| \le \overline{C}\kappa_j^2(\mathcal{T}).$$

In general $\underline{C}$ and $\overline{C}$ in (3.5) and (3.7) could be different.

The element $\mathcal{T}_0$. Let $\mathcal{T}_0 \subset \Omega_j^{(r_j)}$, then $A_j \in \overline{\mathcal{T}}_0$, and we assume that (3.5) holds (but not (3.4)).

Constants $\underline{C}$, $\overline{C}, A, A^*$ in (3.4), (3.5) and (3.7) are independent of $\mathcal{T} \in \mathfrak{M} \in \mathcal{F}$.

2) Elements in $\Omega_0^{(r/2)}$.

Let $\mathcal{T} \in \Omega_0^{(r/2)}$. Then let

(3.8) $$\underline{C} \le \nu(\mathcal{T}) \le \overline{C}$$

with $\underline{C}$, $\overline{C}$ independent of $\mathcal{T} \in \mathfrak{M} \in \mathcal{F}$.

Condition (3.8) shows that there is only finite number of elements in $\Omega_0^{(r/2)}$ (dependent on various constants in (3.1) (3.2), (3.8) but independent of $\mathfrak{M} \in \mathcal{F}$). Elements of $\mathfrak{M}$ are curvilinear with the sizes proportional to the distance from the vertex (except those elements which contain the vertex).

2.4. The meshes and the finite elements.

Let us consider a family $\mathcal{F}$ of meshes on Ω. A mesh $\mathfrak{M} \in \mathcal{F}$ is a partition of Ω into the set of elements $\mathcal{T} \in \mathfrak{M}$ which satisfy conditions given in the Section 2.3 and $\overline{\Omega} = \bigcup_{\mathcal{T}\in\mathfrak{M}} \overline{\mathcal{T}}$. To every $\mathcal{T} \in \mathfrak{M}$ we have associated the analytic map $M_{\mathcal{T}}$. Obviously we can speak about the vertices and sides of $\mathcal{T}$. We will assume that the elements of $\mathcal{T} \in \mathfrak{M}$ satisfy the usual conditions i.e. if $\mathcal{T}_i, \mathcal{T}_j \in \mathfrak{M}$ then either $\overline{\mathcal{T}}_i, \overline{\mathcal{T}}_j$ are disjoint or have common vertex or common sides. If $\overline{\mathcal{T}}_i, \overline{\mathcal{T}}_j$ have common sides then the mapping $M_{\mathcal{T}_i}, M_{\mathcal{T}_j}$ coincides on the common side.

Let us now consider the mesh of elements in $\Omega_j^{(r_j)}$. Because (3.3) and (3.4) we can speak about the layers of the elements. The zero layer consists of all elements which (closure) includes the vertex A_j. Denote by $\mathcal{L}_k$ the k-th layer of elements. Then we define $\mathcal{L}_k$ layer as the set of all elements $\mathcal{T} \subset \Omega_j^{(r_j)}$ such that they do not belong to the layers $\mathcal{L}_j$, j = 1,2,...,k-1 but $\overline{\mathcal{T}} \cap \bigcup_{\mathcal{T}\in\mathcal{L}_{k-1}} \overline{\mathcal{T}} \neq 0$. From the conditions (3.1)-(3.5) we see that there is a constant K_0 dependent only on the constants in (3.1)-(3.5) so that any layer $\mathcal{L}_k$ consists of at most K_0 elements.

We will now consider the family $\mathcal{F}$ of meshes (of the elements described in Section 3.2) which is characterized by the size of the smallest element in the mesh. We will assume for given $\sigma < 1$ and $n > 0$ an integer we have for $\mathcal{T}_0^{(j)} \subset \Omega_j^{(r_j)}$, $A_j \in \overline{\mathcal{T}}_0^{(j)}$

$$\underline{C}\,\sigma^n \le \kappa(\mathcal{T}_0^{(j)}) \le \overline{C}\,\sigma^n. \tag{4.1}$$

Hence the mesh $\mathfrak{M}$ is characterized by the parameters (σ,n) and we will write $\mathfrak{M}(\sigma,n)$. Note that with increasing n, the number of elements in $\Omega_j^{(r_j)}$ is growing, while the number of elements in $\Omega_0^{(r/2)}$ is uniformly bounded. In fact we may assume that the mesh in $\Omega_0^{(r/2)}$ is independent of

n. We show the typical mesh in the neighborhood of a vertex in the Fig. 2.4.1. The layers of the elements are shadowed. In general (σ,n) can depend on j, i.e, can be different in every vertex neighborhood. Nevertheless we will assume that (σ,n) are independent of j.

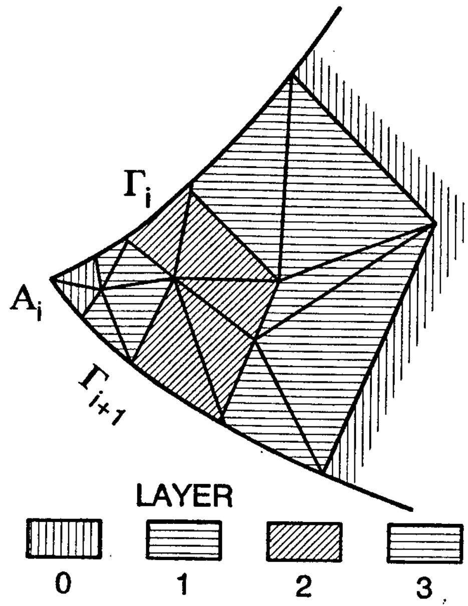

Fig. 2.4.1. Typical mesh with shadowed layers

Mesh of this type will be called a *geometrical mesh* with the factor σ. We can also have $\sigma = 1$. Then the mesh $\mathfrak{M}(\sigma,n)$ will have number of elements independent of n. We can then assume that the mesh is a fixed one. If we know apriori that in the neighborhood of some vertices no singularity will occur, then we select $\sigma = 1$ in these neighborhoods. This is typical for example when we impose the symmetry condition on some part of the boundary.

Remark 4.1. Denote by $\mathcal{T}_k^{(j)}$ the element in the k layer. Then using (3.1) (3.3)(3.4)(3.5) and (4.1) we have

$$(4.2) \qquad \underline{C}\sigma^n\rho_1^{-k} < \kappa(\mathcal{T}_k^{(j)}) < \bar{C}\sigma^n\rho_2^{-k}$$

with $\rho_i < 1$, $\ell - 1,2$ (depending on the constant in (3.1)-(3.5). In practice we mostly construct the meshes such that (3.1)-(3.5) hold together with

(4.3) $$\underline{C}\sigma^{n-\ell} \le \kappa(\mathcal{T}_\ell^{(j)}) \le \overline{C}\sigma^{n-k}$$

i.e. we have

(4.4) $$\rho_i = \sigma, \ i = 1,2$$

Remark 4.2. Denote by $\bar{\kappa}^{(k)}$ resp $\underline{\kappa}^{(k)}$ the maximum resp minimum of $\kappa(\mathcal{T})$ over all elements in the k^{th} layer. Then $\sum \underline{\kappa}_k^{(k)} \le r_j$ and hence number of layers does not exceed Cn whether (4.1) (resp (4.2)) or (4.3) is used.

The shape functions of degree $p(\mathcal{T})$ are as usually given on the master element D or T. On D we will assume that the degree is separately in every variable. For simplicity we will assume that the degrees of the elements are uniform although the nonuniform distribution is more effective, but would have the same rate of convergences.

Finally we will denote by $S(\mathfrak{M},p) = S(\mathfrak{M}(\sigma,n),\ p) = S(\sigma,n,p) \subset H_0^1(\Omega)$ the finite element space under consideration.

2.5. The h-p version of the finite element method

The finite element method for the model problem (2.1) reads:

Find $u_S \in S = S(\mathfrak{M},p) = S(\sigma,n,p)$ such that

(5.1) $$B(u_S,v) = F(v), \qquad \forall\, v \in S(\sigma,n,p),$$

where

(5.2b) $$B(u_S,v) = \int_\Omega \left(\frac{\partial u_S}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u_S}{\partial x_2}\frac{\partial v}{\partial x_2}\right),$$

(5.2b) $$F(u) = \int_\Omega f\, v + \int_{N_\Gamma} g\, v.$$

Then with

$$(B(u,u))^{1/2} = \|u\|_E$$

we have

$$\|u_0 - u_S\|_E = \inf_{\chi\in S} \|u_0 - \chi\|_E$$

where u_0 is the exact solution defined in the Section 2.2. Obviously $\|u\|_E = |u|_1$ where by $|u|_1$ we denoted the seminorm in $H^1(\Omega)$.

Let us now formulate typical theorems. We outline the main ideas of their proof in the Section 2.6. For details we refer to [9], [10], [11].

Theorem 2.5.1. Let u_0 be solution of the problem (2.1). Assume further that for any $x \in \Omega$ and any $\alpha = (\alpha_1,\alpha_2)$, $\alpha_i \geq 0$ integers we have (see (1.2))

$$|D^\alpha u_0| \leq C_0 d_0^\alpha \alpha! . \tag{5.3}$$

Consider now the mesh $\mathfrak{M}(\sigma,n)$ with $\sigma = 1$ and $p(\mathcal{T}) = n$, $\mathcal{T} \in \mathfrak{M}$. Then

$$|u_0 - u_S|_{H^1(\Omega)} = \inf_{\chi\in S(\sigma,n,p)} |u_0 - \chi|_{H^1(\Omega)} \leq C e^{-\gamma n} \tag{5.4a}$$

$$|u_0 - u_S|_{H^1(\Omega)} = \inf_{\chi\in S(\sigma,n,p)} |u_0 - \chi|_{H^1(\Omega)} \leq C e^{-\bar{\gamma} N^{1/2}} \tag{5.4b}$$

where γ, $\bar{\gamma} > 0$ and $C > 0$ depend on the solution, the elements, the domain but are independent of n and N; N is the number of degrees of freedom ($N = \dim S(\sigma,n,p)$).

Remark 5.1. The condition (5.3) is equivalent with the assumption that u is analytic on $\bar{\Omega}$.

Remark 5.2. Because $\sigma = 1$ the number of elements is independent of n. Hence the method is the p-version of the finite element method.

Theorem 2.5.2. Let u_0 be the solution of the problem (2.1). Assume that $u_0 \in \mathcal{B}(\beta)$ as defined in the Section 2.1. Consider now the geometric mesh $\mathfrak{M}(\sigma,n)$ with $\sigma < 1$ and $p(\mathcal{T}) = \mu n$, $\mu > 0$, or properly chosen. Then there exist $C, \gamma, \bar{\gamma} > 0$ such that

$$|u_0 - u_S|_{H^1(\Omega)} = \inf_{\chi \in S(\sigma,n,p)} |u_0 - \chi|_{H^1(\Omega)} \le C\, e^{-\gamma n}, \tag{5.5a}$$

$$|u_0 - u_S|_{H^1(\Omega)} = \inf_{\chi \in S(\sigma,n,p)} |u_0 - \chi|_{H^1(\Omega)} \le C\, e^{-\bar{\gamma} N^{1/3}} \tag{5.5b}$$

where $C, \gamma, \bar{\gamma}$ are independent of n and N.

Remark 5.3. If $\sigma < 1$ then number of elements in the mesh $\mathfrak{M}(\sigma,n)$ is proportional to n, and for $p = \mu n$ (5.5b) follows directly from (5.5a).

In the Theorem 5.2 the assumption $u_0 \in \mathcal{B}(\beta)$ is essential. As we mentioned in the Section 1, this assumption is satisfied practically for every problem in structural mechanics where all the input data are piecewise analytic.

The h-p version simultaneously changes the mesh and the degree of elements. In practice often the geometrical mesh $\mathfrak{M}(\sigma,n)$ is a priori constructed for fixed σ and $n = n_0$ and then p is increased i.e. the space $S(\sigma,n_0,p)$ with p increasing is used. Then we see two phases in the behavior of the method. In the first phase the convergence is very similar as in the h-p version with the exponential convergence and in the second phase for large p the performance is similar as in the p-version with an algebraic rate of convergence.

Remark 5.4. The constants γ, $\bar{\gamma}$ depend among others on σ and β. In [12] we analyzed one dimensional case and found that $\sigma \approx 0.15$ is optimal. In the next section we will see some numerical results related to the question

of the optimal selection of σ.

Remark 5.5. The h-p version for elliptic equations of order 2m was addressed in [13].

Remark 5.6. We formulated theorems 1 and 2 for Laplace equation only. Because the essential feature of these theorems is the approximation, they hold in general setting when $u_0 \in \mathcal{B}(\beta)$, for example for the elasticity problem.

2.6. Numerical experiments

In this section we will show a typical numerical example. Consider the elasticity problem on a cracked domain shown in the Fig. 2.6.1a. Because we will consider the symmetric problem, only the half domain will be considered as show in Fig. 2.6.1b.

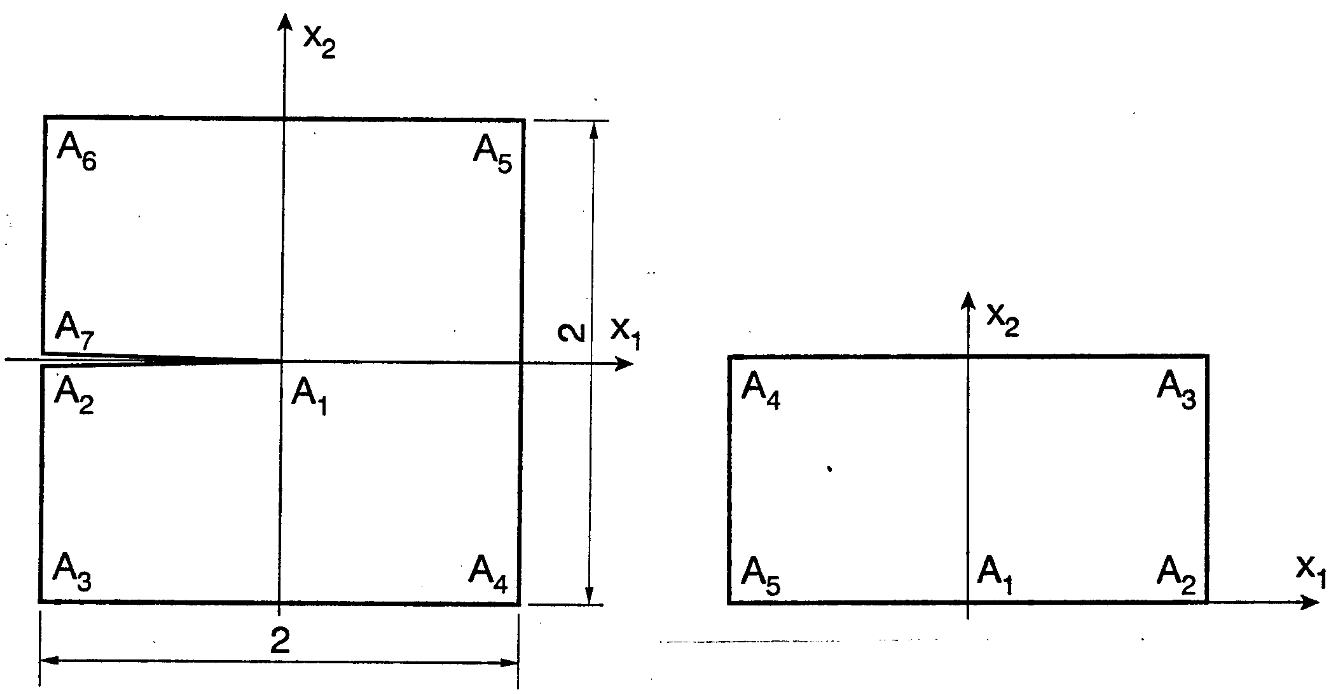

Fig. 2.6.1. The scheme of the cracked domain

a) The cracked domain, b) the symmetric half of the cracked domain.

We will assume that material is isotropic homogeneous with Poisson ratio $\nu = 0.3$. On the boundary Γ of Ω depicted in the Fig. 2.6.1, we impose nonhomogeneous traction (Neumann) conditions so that the exact solution is symmetric stress intensity mode for which the stress behaves as $O(r^{-1/2})$. Hence we have only singularity in the origin in the vertex A_1. Therefore the mesh will be refined only in the neighborhood of the vertex A_1 (i.e. $\sigma = 1$ in the neighborhood of all other vertices). Because of the symmetry we will consider only the problem on the half of the domain as shown in Fig. 2.6.1b. We will use $\sigma = 0.15$.

In the Fig. 2.6.2 we show the sequence of the meshes with n layers (not drawn in scale).

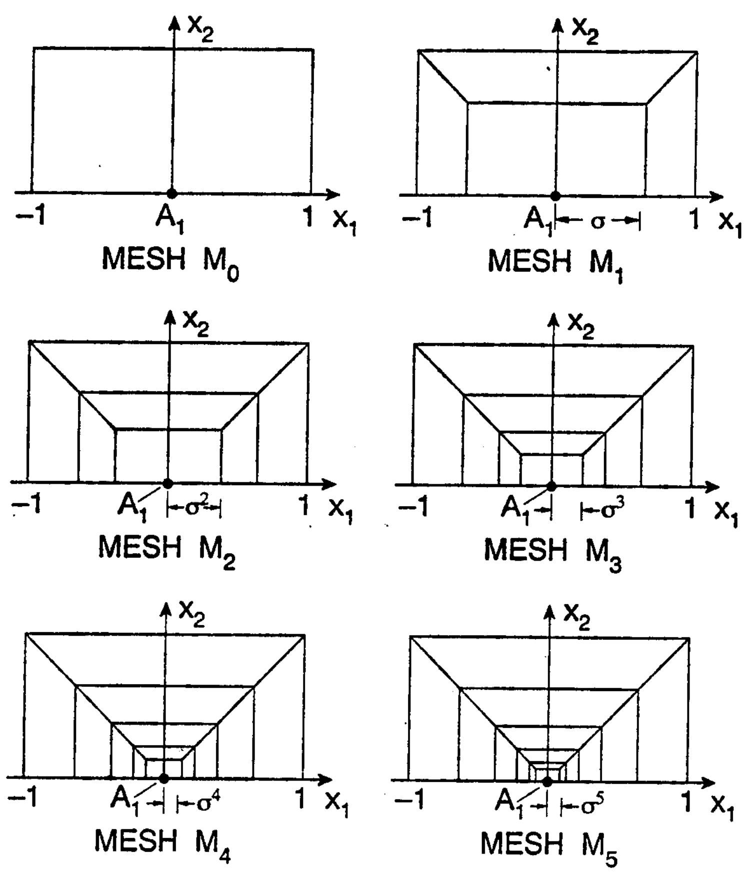

Fig. 2.6.2. The sequence of the geometric meshes $\mathfrak{M}(\sigma,n)$, $n = 0,\ldots,5$.

We will use $p = n + 1$ in our example and uniform p. Based on the Theorem 2.5.2 we expect that the error $\|e^{(n)}\|_E$ is bounded by $C\, e^{-\gamma N^{1/3}}$ with C and γ independent of n and N. We will assume that

$$\|e^{(n)}\|_{ER} = \frac{\|u_0 - u_{FE}^{(n)}\|_E}{\|u_0\|_E} \approx C(n)e^{-\gamma N^{1/3}} \tag{6.1}$$

and determine the value γ from the two successive values of $\|e^{(n)}\|_{ER}$. Then we compute for every n the value C and γ. The results are given in the Table 2.6.1. Although Theorem 2.5.2 gives only an upper estimate it seems that $\gamma(n)$ and $C(n)$ converge as $n \to \infty$. This is not surprising in our particular case, but it cannot be concluded neither from the theorem or the theory presented here.

Table 2.6.1. Performance of the h-p version

n	p	N	$N^{1/3}$	$\|e^{(n)}\|_{ER}\%$	$\gamma(n)$	$C(n)$
0	1	9	2.08	60.92	0.741	1.455
1	2	48	3.63	20.23	0.740	1.455
2	3	121	4.95	7.61	0.776	2.098
3	4	256	6.35	2.57	0.720	1.810
4	5	477	7.82	0.90	0.670	1.683
5	6	808	9.31	0.33	0.670	1.688

In the Fig. 2.6.3 we show the error $\|e^{(n)}\|_{ER}$ as function of N. We plot the graph in $N^{1/3} \times \lg \|e^{(n)}\|_{ER}\ \%$ scale. Then the graph would be the straight line if the error would obey exactly (6.1).

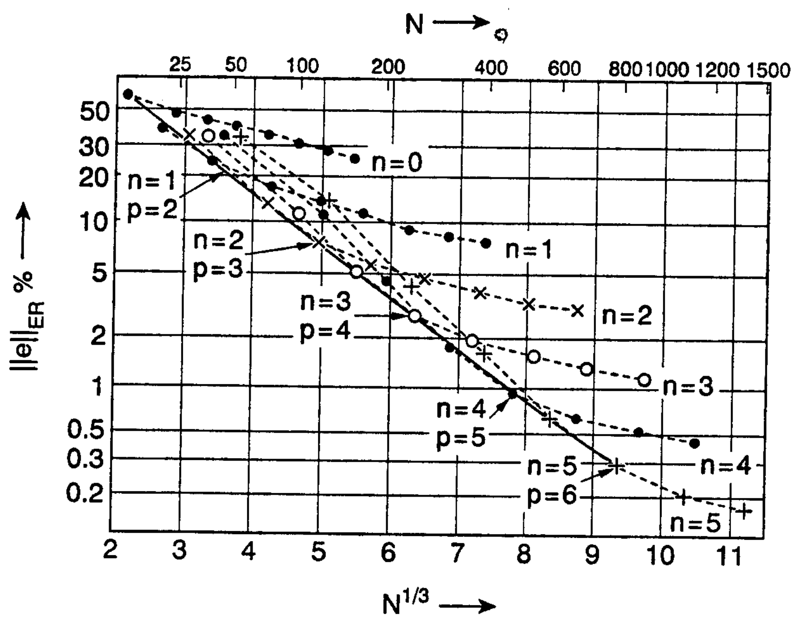

Fig. 2.6.3. Performance of the p and h-p version for $S(\sigma,n,p)$, $\sigma = 0.15$

In the Fig. 2.6.3 we also show the error of the p-version i.e. the error using S(0.15, n, p) for fixed n and $1 \le p \le 8$. The h-p version graph connects the points giving the accuracy for $p = n + 1$. We see that we obtain a straight line. Table 2.6.1 and Fig. 2.6.3 shows that the (6.1) gives not only the upper estimates but also describes well the behavior of the error. From the Fig. 2.6.3 we also see that for given n the error behavior as function of p can be divided into two phases. In the first phase when $p \le n + 1$ we see exponential convergence while for $p \ge n + 1$ we see only an algebraic one.

The performance depends among others on the value of σ and β. In one dimensional setting $\sigma \approx 0.15$ is optimal, see [12]. In the Fig. 2.6.4 we show the performance of the h-p version for $p = n + 1$ as function of σ. The Fig. 2.6.4 is drawn in the same scale as Fig. 2.6.3. We see that the value $\sigma = 0.15$ gives the optimal results.

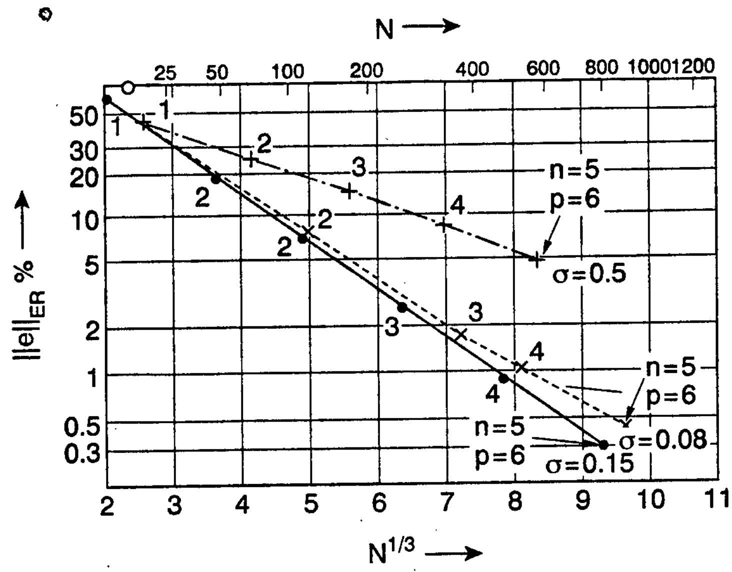

Fig. 2.6.4. The performance of the h-p version for various σ

Remark 6.1. We have shown the performance of the h-p version as function of N. Of course for higher p the stiffness matrix is more dense and the cost of construction of the stiffness matrix is higher too. Hence the right performance description will be graph of the computer cost $\times$ accuracy. This problem was addressed in [14] [15].

Remark 6.2. The meshes for the h-p version have a special character. The usual available mesh generators are not producing the appropriate meshes for the h-p version yet.

2.7. Outline of the theory.

In this section we will briefly outline the theory leading to the theorem 2.5.1 and theorem 2.5.2. The main ideas in two dimensional and three dimensional case are similar, but technicalities are much more difficult in 3 dimensions.

1) The regularity problem.

The first major problem is to characterize the space of solutions under consideration. The space should be, on one hand, as small as possible, to give the possibility to employ its special properties, but on the other hand the space has to be large enough that it would cover most problems in engineering practice. As was mentioned earlier typical engineering problems are characterized by piecewise-analytic input data. Hence we need to describe the spaces of the solutions of such problems.

There is a large literature about the regularity of the solution of elliptic problems. Nevertheless those theories are not directed to the goals of numerical solutions. In a series of papers, [4] [5] [6] [7] we developed the theory which characterizes the regularity of the solutions in the terms of countably normed spaces (see (1.1), (1.2)) which is very advantageous for the analysis of the h-p version. This theory in the cited papers encompasses larger class than the one used here.

We do partition the domain in the (overlapping) areas and characterize the regularity in these areas in a special way. In two dimensional settings we distinguish between vertex neighborhood (denoted above by $\Omega_j^{(r_j)}$) and the internal domain (denoted by $\Omega_0^{(r/2)}$). In 3 dimensions we have considered four regions separately.

2) Approximation of functions defined on the standard square $D = I^2$, $I = (-1,1)$.

The major theorem is

<u>Theorem</u> 2.7.1. Let $D^{\alpha}u \in L^2(D)$, $\alpha = (\alpha_1,\alpha_2)$, $0 \le \alpha_\ell \le t_\ell + 1$ $t_\ell \ge 0$, $\ell = 1,2$. Then there exists a polynomial $P(x_1,x_2)$ of degree t_1,t_2 in x_1,x_2 such that for $0 \le \alpha_i \le 1$ and $1 \le s_\ell \le t_\ell$, $\ell = 1,2$ we have

$$\|D^{\alpha}(u-P)\|^2_{L^2(D)} \le C\left[\frac{(t_1-s_1)!}{(t_1+s_1+2(1-\alpha_1))!}\sum_{0\le\beta_2\le1}\|D^{s_1+1,\beta_2}u\|^2_{L^2(D)}\right.$$

$$\left.+\frac{(t_2-s_2)!}{(t_2+s_2+2(1-\alpha_2))!}\sum_{0\le\beta_1\le1}\|D^{\beta_1,s_2+1}u\|^2_{L^2(D)}\right]. \tag{7.1}$$

For $0 \le \alpha_2 \le 1$, $\alpha_1 = 0$

$$\|D^{\alpha}(u-P)(\pm1,x_2)\|^2_{L^2(I)} \le C\left[\frac{(t_2-s_2)!}{(t_2+s_2+2(1-\alpha_2))!}\sum_{0\le\beta_1\le1}\|D^{\beta_1,s_2+1}u\|^2_{L^2(D)}\right]. \tag{7.2}$$

For $0 \le \alpha_1 \le 1$, $\alpha_2 = 0$

$$\|D^{\alpha}(u-P)(x_1,\pm1)\|^2_{L^2(I)} \le C\left[\frac{(t_1-s_1)!}{(t_1+s_1+2(1-\alpha_1))!}\sum_{0\le\beta_2\le1}\|D^{s_1+1,\beta_2}u\|^2_{L^2(D)}\right]. \tag{7.3}$$

$$(u-P)\ (\pm1,\ \pm1) = 0 \tag{7.4}$$

We denoted $D^{\alpha_1,\alpha_2}u = \dfrac{\partial^{\alpha_1+\alpha_2}u}{\partial x_1^{\alpha_1}\ \partial x_2^{\alpha_2}}$ in (7.1) - (7.3) and thereafter.

We see that smoothness of the trace of the function $(u - P)$ and its norm on the sides of D is estimated by various derivatives of u in D. As a simple corollary from the theorem 7.1 we get

<u>Theorem</u> 2.7.2. Let $\psi(x) = (u - P(x_1,x_2))$, $x_1 = x_2 = x \in I$ then for $|\alpha| = 1$, (i.e. x lies on the diagonal of D),

$$\|D^{\alpha}\psi\|^2_{L^2(I)} \le C\left[\frac{(t_1-s_1)!}{(t_1+s_1)!}\sum_{0\le\alpha_2\le 1}\|D^{s_1+1,\alpha_2}u\|^2_{L^2(D)} + \right. \tag{7.5}$$

$$\left.\frac{(t_2-s_2)!}{(t_2+s_2)!}\sum_{0\le\alpha_1\le 1}\|D^{\alpha_1,s_2+1}u\|^2_{L^2(D)}\right]$$

The constant C in Theorem 2.7.1 and 2.7.2 is independent of t,s. For the proof in the 3 dimensional setting we refer to [16].

3) Regularity of the transformed function on the standard square

As said above, there is a mapping $M_{\mathcal{T}}$ which maps D onto $\mathcal{T}$. We assumed that the mapping $M_{\mathcal{T}}$ has properties given in (3.1) (3.2). We also have assumed that the solution $u \in \mathcal{B}(\beta)$. Consider now the geometrical mesh $\mathfrak{M}(\sigma,n)$ and let $\mathcal{T} \in \Omega_j^{(r_j)}$ and $\overline{\mathcal{T}} \cap A_j = 0$. Then for $x \in \mathcal{T}$ we have from (1.1)

$$|D^{\alpha}u|(x) \le C\, d^{\alpha}\, \alpha!\,(\kappa)^{-(\beta+\alpha_1+\alpha_2-1)}$$

$$= C\left(\frac{d_1}{\kappa}\right)^{\alpha_1}\left(\frac{d_2}{\kappa}\right)^{\alpha_2}\alpha!\ \kappa^{1-\beta}, \tag{7.6}$$

where we wrote κ instead of κ_j defined in (3.3), d_1,d_2 instead $d_j^{(1)}$, $d_j^{(2)}$, and β, C instead of β_j, C_j.

Let now

$$U(\eta) = u(M_{\mathcal{T}}(\eta)), \tag{7.7}$$

then $U(\eta)$ is defined on D and we have

Theorem 2.7.3. For any integer $s \ge 0$ and any $\eta \in D$ we have

For $0 \le \alpha_2 \le 1$

$$|D^{s,\alpha_2}U(\eta)| \le C\, d^s s!\ \kappa^{1-\beta}. \tag{7.8}$$

For $0 \le \alpha_1 \le 1$

$$(7.9) \qquad |D^{\alpha_1,s}\, U(\eta)\ | \le C\, d^s\, s!\ \kappa^{1-\beta},$$

where the constants C,d are independent of η and s.

Quite similar result holds for $\mathcal{T} \in \Omega_0^{(r/2)}$. The elements $\mathcal{T}_0$ for which $\overline{\mathcal{T}}_0 \cap A_j \neq 0$ have to be treated separately as shown below.

4) The approximation on the element $\mathcal{T}$

We use now the estimates (7.8), (7.9) together with the results of the Theorem 2.7.1 and obtain

<u>Theorem</u> 2.7.4. There exists a polynomial $P(\eta_1,\eta_2)$ of the degree p in η_1 and η_2 such that for $0 \le \alpha_i \le 1$, $i = 1,2$

$$(7.9) \qquad \|D^{\alpha}(U-P)\|^2_{L^2(D)} \le C\left\{\kappa^{2(1-\beta)}\sum_{\ell=1}^{2} F(d)^p p^{3-2(1-\alpha_\ell)}\right\},$$

$$(7.10a) \qquad \|D^{\alpha_2}(U-P)(\pm1,\eta_2)\|^2_{L^2(I)} \le C\left\{\kappa^{2(1-\beta)}(F(d))^p p^{3-2(1-\alpha_2)}\right\},$$

$$(7.10b) \qquad \|D^{\alpha_1}(U-P)(\eta_1,\pm1)\|^2_{L^2(I)} \le C\left\{\kappa^{2(1-\beta)}(F(d))^p p^{3-2(1-\alpha_1)}\right\},$$

$$(7.11) \qquad (U-P)(\pm1,\ \pm1) = 0\ .$$

Function $F(d)$ is increasing but for any d, $F(d) < 1$. The function $F(d)$ is obtained from (7.1) by the optimal choice of s for given degree of the polynomial. For detail properties of this function we refer to [9], [10]. From (7.9)-(7.11) we see that the convergence rate in p is exponential.

Using now the transformation $M_{\mathcal{T}}^{-1}$ and utilizing (3.2) we get

<u>Theorem</u> 2.7.5. Let $\varphi = M_{\mathcal{T}}^{-1}P$, where P is the polynomial in the Theorem 2.7.4. Then

$$\left\|\frac{\partial}{\partial x_i}(u-\varphi)\right\|^2_{L^2(\mathcal{T})} \le C\sum_{|\alpha|=1}\|D^\alpha(U-P)\|^2_{L^2(D)}, \quad i=1,2$$

(7.12)

$$\le C\kappa^{2(1-\beta)}Q^p$$

where $0 < Q < 1$. It is independent on κ, but depends on the constants in (3.1) and (3.2)

The elements which have a vertex in the A_j have to be treated separately. Using the assumed properties of u it is easy to obtain

Theorem 2.7.6. Let $\mathcal{T}_0 \in \mathfrak{M}$ is such that $\overline{\mathcal{T}}_0 \cap A_j \ne 0$. Then we have

$$\left\|\frac{\partial}{\partial x_i}(\varphi-u)\right\|^2_{L^2(\mathcal{T}_0)} \le C\kappa^{2(1-\beta)}, \quad i=1,2 \tag{7.13}$$

Function φ is here mapped bilinear function.

5) The adjustments of local approximation

The results in 4) allow to estimate the approximation by a polynomial in every element separately. Nevertheless this construction of the approximation does not lead to the function which is continuous across the boundaries of the elements i.e. conforming elements. The continuity is guaranteed only in the vertices of the elements because the approximation coincides here with the approximated function. Hence we have to make corrections which will delete these (sides) discontinuities . This is made as follows. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ are two elements with common edge ℓ and let $\ell \cap \Gamma = \emptyset$. Assume that S_1 and S_2 are the sides of the standard square D which by $M_{\mathcal{T}_1}$ and $M_{\mathcal{T}_2}$ are mapped on the edge ℓ.

Without any loss of generality we will assume that $S_1 = S_2 = S$. Let U_i, P_i, $i = 1,2$ be the mapped functions U and the polynomial P considered in the Theorem 2.7.4. Then obviously $U_1 = U_2$ on S and $\psi = P_1 - P_2$ is a

polynomial of degree p on S and $\psi = 0$ in the end points of S. Using (7.10 a) we get with $0 < Q < 1$

$$\|\psi\|^2_{H^1(S)} \le C(\kappa_1^{2(1-\beta)} + \kappa_2^{2(1-\beta)})Q^p. \tag{7.14}$$

Without any loss of generality we will assume $\kappa_1 \ge \kappa_2$ and construct on D a polynomial Q_1 of degree p (in η_1 and η_2) such that $Q_1 = \psi$ on S and $Q_1 = 0$ on $\partial D - S$ and

$$\|Q_1\|_{H^1(D)} \le C\|\psi\|_{H^1(S)}. \tag{7.15}$$

We use now instead P_1 the polynomial $P_1 - Q_1$ and get

$$\begin{aligned} \|U_1-(P_1-Q_1)\|_{H^1(D)} &\le \|U_1-P_1\|_{H^1(D)} + \|Q_1\|_{H^1(D)} \\ &\le C(\|U_1-P_1\|_{H^1(D)} + \|\psi\|_{H^1(S)}) \\ &\le C(\kappa_1^{2(1-\beta)}\; Q^p)^{1/2}. \end{aligned} \tag{7.16}$$

Hence the error of the finite element method can be estimated by the errors in the single elements as described in Theorem 7.5 and 7.6.

6) Proof of the Theorem 2.5.1, and 2.5.2.

Let us first address Theorem 2.5.1. By the assumption the mesh $\mathfrak{M}$ has $m(\sigma,n)$ elements (independent of m) and u satisfies (5.3). Using the results in the previous paragraphs we construct function $\varphi \in S(\sigma,n,p)$, $\sigma = 1$ such that

$$\|u - \varphi\|^2_E \le Cm\, Z^p = Cm\, e^{-2\alpha p}, \quad \alpha > 0, \quad 0 < Z < 1,$$

which yields (5.4a). Because the number of degrees of freedom N (= dimension of $S(\sigma,n,p)$) is of order mp^2 we obtain (5.4b).

Let us now address Theorem 2.5.2. We divide the elements of the geometrical mesh in three groups. Elements in $\Omega_0^{(r/2)}$, elements in $\Omega_j^{(r_j)}$

$j = 1,\dots.M$ but such that $\bar{\mathcal{T}} \cap A_j = \emptyset$ and $\mathcal{T} \subset \Omega_j^{(r_j)}$, $\bar{\mathcal{T}} \cap A_j \neq \emptyset$. Obviously we have at most CM elements (with M being the number of vertices) of the third group. Using Theorem 2.7.6 the error (square) on these elements in the third group is estimated by $CM(\sigma^n)^{2(1-\bar{\beta})}$ where $\bar{\beta} = \max \beta_i$. In the second group in every $\Omega_j^{(r_j)}$ we have at most Cn layers and the number of elements in every layer is uniformly bounded (see Sec. 2). Using Theorem 2.7.5 we see that the error (square) in the domain $\Omega_j^{(r_j)}$ can be bounded by

$$C \sum_{j=1}^{n} \sigma^{2n(1-\beta_j)} \rho_2^{-2(1-\beta)_j} Q^p \le$$

$$\le C\sigma^{2n(1-\bar{\beta})} \rho^{-2n(1-\bar{\beta})} Q^p$$

where $Q < 1$ is independent of p and σ. Hence the total error (square) in the entire Ω (composed from the errors of the three mentioned graphs) is for $p = \mu n$ bounded by

$$C\left[Q^{\mu n} + \sigma^{2n(1-\bar{\beta})} + \left(Q^{\mu}\left(\frac{\sigma}{\rho_2}\right)^{2(1-\bar{\beta})}\right)^n\right] \le CZ^p,\ Z < 1$$

provided that μ was properly chosen depending on σ, ρ, $\bar{\beta}$.

Obviously the number of elements in $\mathfrak{M}$ is bounded by Cn and we get (5.5a), (5.5b) then follows from the fact that the dimension of the space of polynomials of degree p in $\mathcal{T}$ is p^2.

So far we have assumed that the elements are quadrilateral. If they are triangular then we use first the approximation on the square D and the only difference is in the adjusting phase. Here we use Theorem 2.7.2 for the estimate on the diagonal of the square D. The extension from the sides of the master triangle inside it is standard. Of course we have to see that at the diagonal of D the function is the polynomial of degree $2p$, but this influences the constants but not the rate in the bounds.

3. The h-p version in three dimensions

In this section we will formulate the results for the 3 dimensional problem which are analogous to those mentioned in Section 2. We will underline similarities and differences between the two and three dimensional results.

3.1. Preliminaries

Because of simplicity we will consider only polyhedral domains although our results have much more general character.

Figure 3.1.1 shows a typical domain. The planes KLNO and MLNO coincide (crack type). By A,B... we denoted the *vertices* of the domain. The *edges* of the domain Ω are straight lines with the vertices at their end. For example $\overline{AB}$ is an edge. We have in the Figure 3.1.1 only edges where the integral angle $\neq \pi$. Nevertheless for the same reasons as in 2 dimensional case we can or have to add additional vertices or edges, e.g. the edge BL etc. A set of edges creates the boundary of a face, which is a polygon. For example B C H I J K L M G are the vertices of a face. As in the two dimensional case we will assume that on every face a boundary condition is of *the same type*. In our case we have only plane faces. Nevertheless the edges and faces can be curved too.

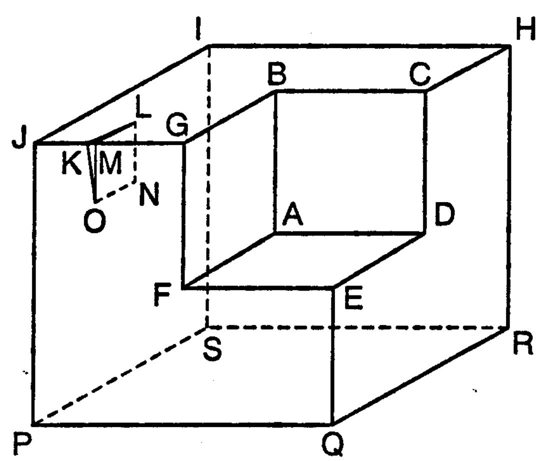

Fig. 3.1.1. The polyhedral domain

In the two dimensional case--in the Section 2.1--we partitioned the domain in the neighborhood of the vertices and rest of the domain. The three dimensional case is more complex. Here we have to partition the domain in the following types of regions.

1) Neighborhood of the edges (not close to the vertices).

2) Neighborhood of the vertices edges (close to the vertices and edges).

3) Neighborhood of the vertices (not close to the edges).

4) The regular region.

Let us describe these regions more precisely.

1) The neighborhood of the edge $\Omega_{e}^{R,\delta}$ (not close to a vertex).

Assume that the edge e under consideration is

$$e = \{x_1, x_2, x_3 \mid x_1 = x_2 = 0,\ 0 < x_3 < 1\}. \tag{1.1}$$

Then we denote for $R > 0$, $\delta > 0$

$$\Omega_{e}^{R,\mathcal{T}} = \{(x_1, x_2, x_3) \in \Omega \mid x_1^2 + x_2^2 = r < R \\ \delta < x_3 < 1 - \delta\} \tag{1.2}$$

and assume that R, δ are sufficiently small.

Let us consider the edge AB in Figure 3.1.1 as an example, Fig. 3.1.2 shows the region $\Omega_{e}^{R,\delta}$.

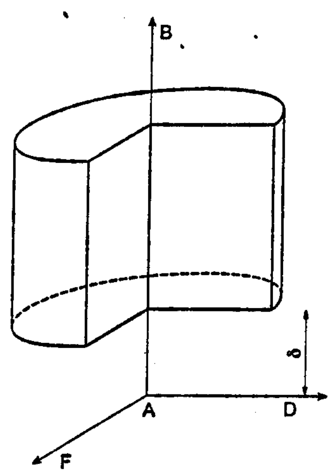

Fig. 3.1.2. The edge neighborhood

Obviously the domain $\Omega_{e}^{R,\delta}$ can be described well in the cylindrical coordinates.

2) The neighborhood of the vertex A and the edge e (the vertex-edge neighborhood). Assume once more that the edge is given as in (1.1)

Let $x = (x_1,x_2,x_3)$, $\rho^2(x) = x_1^2 + x_2^2 + x_3^2$, $r^2(x) = x_1^2 + x_2^2$, $\sin\varphi = r/\rho$ and denote

$$\Omega_{e,A}^{R,\Phi} = \{x \in \Omega \mid \rho(x) \le R,\ \sin\varphi < \sin\Phi\}$$

with R and Φ sufficiently small.

The domain $\Omega_{e,A}^{R,\Phi}$ associated to the edge $AB = e$ and vertex A is shown in the Figure 3.1.3.

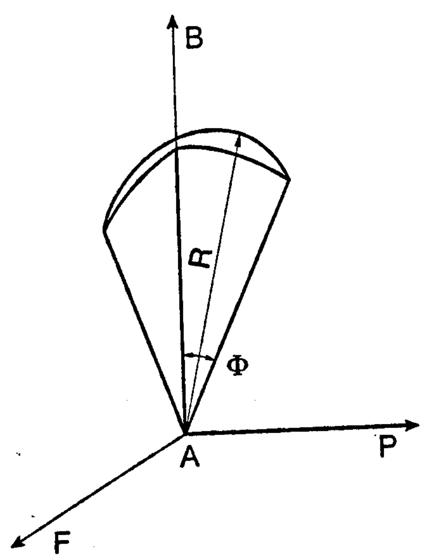

Fig. 3.1.3. The vertex-edge domain

Obviously the domain $\Omega_{e,A}^{R,\Phi}$ can be described well in the spherical coordinates.

3) The neighborhood of the vertex (not close to the edges).

Assume now that in the vertex A is the end of the edges $e_1,\ldots,e_n$.

For example in the Fig. 3.1.1 vertex A is the end of the edges $e_1 = \overline{AB}$, $e_2 = \overline{AD}$, $e_3 = \overline{AF}$. Denote now by $\Omega^{R,\Phi_i}_{e_i,A}$ the edge-vertex domain introduced above. We will assume that Φ_i are sufficiently small that $\Omega^{R,\Phi_i}_{e_i,A} \cap \Omega^{R,\Phi_j}_{e_j,A} = \emptyset$ for $i \neq j$. Then we define

$$\Omega^R_A = \{x \in \Omega,\ \rho(x) < R,\ x \notin \Omega^{R,\Phi_i}_{e_i,A},\ i = 1,\dots,n\}.$$

4) The regular region.

For the given domain we defined the domains $\Omega^{R,\delta}_e$, $\Omega^{R,\Phi_i}_{e_i,A}$, Ω^R_A. Then we

$$\text{define}\quad \Omega^R_0 = \Omega - \bigcup_e \bar{\Omega}^{R_i\delta_i}_{e_i} - \bigcup_{e,A} \bar{\Omega}^{R,\Phi_i}_{e_i,A} - \bigcup_A \bar{\Omega}^R_A .$$

Let us now make the comments about selection of the parameters in the 4 particular domains we have introduced. We will assume that

1) The domains of <u>one</u> category (i.e. edge or edge-vertex, or vertex domains) do not intersect. On the other hand we will assume that the parameters are selected to that $\Omega^R_0 \cup \bigcup_i \Omega^{R_i\delta_i}_{e_i} \cup \bigcup_{i,A} \Omega^{R,\Phi_i}_{e_i,A} \cup \bigcup_A \Omega^R_A = \Omega$. (Note that the values R are different for different regions). The regions of different categories may intersect (i.e. partially overlap) so that any <u>entire</u> element $\mathcal{T}$ of the used mesh will be in one (or more) domains with one exception. Element having the vertex coinciding with the vertex of the domain Ω -- a "vertex element" has not to be entirely in any one region we introduced. This exception is made for practical reasons related to the mesh generator. We see that the domain Ω^R_0 is analogous to the domain $\Omega^{(r)}_0$ introduced in the Section 2.1.

3.2. The spaces and the model problem

Analogously as in the Section 2.1 we introduce the spaces of functions which norm is different on every neighborhood introduced in Section 3.1.

1) *The neighborhood of the edge* e *(not close to the vertex); the*

For example in the Fig. 3.1.1 vertex A is the end of the edges $e_1 = \overline{AB}$, $e_2 = \overline{AD}$, $e_3 = \widehat{AF}$. Denote now by $\Omega^{R,\Phi_i}_{e_i,A}$ the edge-vertex domain introduced above. We will assume that Φ_i are sufficiently small that $\Omega^{R,\Phi_i}_{e_i,A} \cap \Omega^{R,\Phi_j}_{e_j,A} = \emptyset$ for $i \neq j$. Then we define

$$\Omega^R_A = \{x \in \Omega,\ \rho(x) < R,\ x \notin \Omega^{R,\Phi_i}_{e_i,A},\ i = 1,\dots,n\}.$$

4) The regular region.

For the given domain we defined the domains $\Omega^{R,\delta}_e$, $\Omega^{R,\Phi_i}_{e_i,A}$, Ω^R_A. Then we

$$\text{define } \Omega^R_0 = \Omega - \bigcup_e \bar{\Omega}^{R_i\delta_i}_{e_i} - \bigcup_{e,A} \bar{\Omega}^{R,\Phi_i}_{e_i,A} - \bigcup_A \bar{\Omega}^R_A.$$

Let us now make the comments about selection of the parameters in the 4 particular domains we have introduced. We will assume that

1) The domains of <u>one</u> category (i.e. edge or edge-vertex, or vertex domains) do not intersect. On the other hand we will assume that the parameters are selected to that $\Omega^R_0 \cup \bigcup_i \Omega^{R_i\delta_i}_{e_i} \cup \bigcup_{i,A} \Omega^{R,\Phi_i}_{e_i,A} \cup \bigcup_A \Omega^R_A = \Omega$. (Note that the values R are different for different regions). The regions of different categories may intersect (i.e. partially overlap) so that any <u>entire</u> element $\mathcal{T}$ of the used mesh will be in one (or more) domains with one exception. Element having the vertex coinciding with the vertex of the domain Ω -- a "vertex element" has not to be entirely in any one region we introduced. This exception is made for practical reasons related to the mesh generator. We see that the domain Ω^R_0 is analogous to the domain $\Omega^{(r)}_0$ introduced in the Section 2.1.

3.2. The spaces and the model problem

Analogously as in the Section 2.1 we introduce the spaces of functions which norm is different on every neighborhood introduced in Section 3.1.

1) *The neighborhood of the edge* e *(not close to the vertex); the*

neighborhood $\Omega_{e}^{R,\delta}$. As before we assume that the edge e is given in (1.1) and $\Omega_{e}^{R,\delta}$ by (1.2). For given β, $0 < \beta < 1$ we define the space $\mathfrak{L}_{\beta}(\Omega_{e}^{R,\delta})$ of functions u on $\Omega_{e}^{R,\delta}$ such that

i) u is continuous on $\bar{\Omega}_{e}^{R,\delta}$

ii) for any $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_i \geq 0$ integers

$$|D^{\alpha}(u(x) - u(0,0,x_3)| \leq C\, d^{\alpha}|r(x)|^{-(\beta+\alpha_1+\alpha_2-1)}\alpha! \tag{2.1}$$

$$\left|\frac{d^{\alpha_3}}{dx_3{}^{\alpha_3}}\, u(0,0,x_3)\right| \leq C\, d_3^{\alpha_3}\alpha_3! \tag{2.2}$$

where we denoted

$$d = (d_1,d_2,d_3),\ d_i > 1,\ d^{\alpha} = d_1^{\alpha_1}d_2^{\alpha_2}d_3^{\alpha_3}$$

$$\alpha! = \alpha_1!\ \alpha_2!\ \alpha_3!,\ 0! = 1,\ |\alpha| = |\alpha_1+\alpha_2+\alpha_3|,$$

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$$

and in (2.1) and (2.2) the constant C is independent of α.

2) *The neighborhood* $\Omega_{e_i,A}^{R,\Phi_i}$, *(the vertex-edge neighborhood).*

Let $x \in (x_1,x_2,x_3)$, $\rho^2(x) = x_1^2 + x_2^2 + x_3^2$, $r^2(x) = x_1^2 + x_2^2$, $\sin\varphi = r/\rho$. Let $0 < \beta_1 < 1/2$, $0 < \beta_2 < 1$. Then by $\mathfrak{L}_{\beta_1,\beta_1}(\Omega_{e,A}^{R,\Phi})$ we denote the space of all functions u on $\Omega_{e,A}^{R,\Phi}$ such that

i) u is continuous on $\bar{\Omega}_{e,A}^{R,\Phi}$

ii) for any $\alpha = (\alpha_1,\alpha_2\ \alpha_3)$, $0 \leq \alpha_i$ integers

$$\begin{aligned}|D^{\alpha}(u(x) - u(0,0,x_3)| &\leq \\ &\leq C\,\rho^{-(\beta_1+|\alpha|-1/2)}(\sin\varphi)^{-(\beta_2+\alpha_1+\alpha_2-1)}d^{\alpha}\alpha!\end{aligned} \tag{2.3}$$

$$(2.4)\qquad \left|\frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}}(u(0,0,x_3) - u(0,0,0)\right| \le C\,\rho^{-(\beta_1+\alpha_3-1/2)} d_3^{\alpha_3}\alpha_3!$$

where C is independent of α.

3) *The neighborhood* Ω_A^R *of the vertex* A *(not close to the edge).*

Let $0 < \beta < 1/2$ then by $\mathfrak{L}_\beta(\Omega_A^R)$ we denote the space of all functions u on Ω_A^R such that

i) u is continuous on $\bar{\Omega}_A^R$

ii) for any $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $0 \le \alpha_i$ integers

$$(2.5)\qquad |D^\alpha(u(x) - u(0,0,0)| \le C\,\rho^{-(\beta+|\alpha|-1/2)} d^\alpha \alpha!$$

and C is independent of α.

4) *The regular region* Ω_0^R.

By $\mathfrak{L}_\beta(\Omega_0^R)$ we denote the space of all functions u on Ω_0^R such that

i) u is continuous on $\bar{\Omega}_0^R$

ii) for any $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $0 \le \alpha_i$ integers

$$(2.6)\qquad |D^\alpha u| \le C\,d^\alpha \alpha!$$

and C is independent of α.

As was said above we assume that the regions are overlaping. We define the space $\mathfrak{L}_\beta(\Omega)$ as the space of all u defined on Ω such that after constraining them on the regions $\Omega_e^{R,\delta}$, $\Omega_{e,A}^{R,\Phi}$, Ω_A^R and Ω_0^R they will belong to $\mathfrak{L}_\beta(\Omega_e^{R,\delta})$, $\mathfrak{L}_{\beta_1,\beta_2}(\Omega_{e,A}^{R,\Phi})$, $\mathfrak{L}_\beta(\Omega_A^R)$ and $\mathfrak{L}(\Omega_0^R)$. (We used R which are different in mentioned neighborhoods.)

Analogously as in Section 1 we will consider the problem

$$(2.7a)\qquad -\Delta u = f \quad \text{on } \Omega,$$

$$(2.7b)\qquad u = 0 \quad \text{on } {}^D\Gamma,$$

$$(2.7c)\qquad \frac{\partial u}{\partial n} = g \quad \text{on } {}^N\Gamma.$$

We will understand the problem (2.7) in the weak sense. Find $u \in H_0^1(\Omega) = \{u \in H^1(\Omega),\ u = 0 \text{ on } {}^D\Gamma\}$ such that for any $v \in H_0^1(\Omega)$

$$B(u,v) = \int_\Omega \sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} = \int_\Omega f\, v + \int_{{}^N\Gamma} g\, v$$

holds.

We will assume that on every face of the boundary of Ω either $u = 0$ or $\frac{\partial u}{\partial n} = g$ and g is analytic on the (closed) face where it is defined. Further we will assume that f is analytic on $\bar{\Omega}$. In this case we will speak about the problem (2.7) with analytic input data. Then we have proven in [18]

<u>Theorem</u> 2.1. Let u be the solution of the problem (2.7) with analytic input data. Then $u \in \mathcal{L}_\beta(\Omega)$ for properly selected β.

We see that in 3 dimensions the problem and the description of the regularity of the solution is essentially similar as in two dimensions but much more complex.

3.3 The elements

We will assume that the domain Ω is covered by the mesh $\mathfrak{M}$ which partitions the domain Ω into element $\mathcal{T}$ and $\mathfrak{M} = \{\mathcal{T}\}$ where $\mathcal{T}$ is an element. We will now define elements and their properties in various regions we have defined in Section 3.2. For simplicity, we consider only elements of curved "brick" type i.e. elements which are images of a standard cube. In a similar way as in Section 2 the results hold for tetrahedral which are the images of a part of the standard cube. Let $D = I^3$, $I = (-1,1)$ be the standard cube with local coordinates $\eta = (\eta_1, \eta_2, \eta_3)$, $i = 1,2,3$.

Now we describe the elements in the various regions introduced in Section 2.3.

1) The elements in the edge neighborhood $\Omega_{e}^{R,\delta}$.

We will consider two kinds of elements, $\mathcal{T}$ and $\mathcal{T}_{e}$ separately.

The element $\mathcal{T}$. Let $\overline{\mathcal{T}} \subset \Omega_{e}^{R,\delta}$ (i.e. distance of $\mathcal{T}$ to the edge e is positive) be the image of $\overline{D}$, $D = I^3$, $I = (-1,1)$ 1 by a mapping $M_{\mathcal{T}}$. Assume that $M_{\mathcal{T}}$ is a one to one mapping of $\overline{D}$ onto $\overline{\mathcal{T}}$ with

$$\mathcal{T} = \{x \mid x_i = X_i(\eta),\ \eta \in D,\ i = 1,2,3\}$$

and $X_i(\eta)$ are analytic functions on $\overline{D}$.

Denote

(3.1a) $$\underline{\kappa}(\mathcal{T}) = \min_{x\in\overline{\mathcal{T}}} r(x),$$

(3.1b) $$\kappa(\mathcal{T}) = \max_{x\in\overline{\mathcal{T}}} r(x).$$

We assume about $\mathcal{T}$ and M the following:

i) There exists constant $1 < A < \infty$ such that

(3.2a) $$\frac{\kappa(\mathcal{T})}{\underline{\kappa}(\mathcal{T})} \le A$$

ii) For $|\alpha| = m$, $m > 0$, m an integer and any $\eta \in D$

(3.2b) $$|D^{\alpha}X_i(\eta)| \le C_0 d_0^m\, m!\, \kappa,\ i = 1,2,$$

(3.2c) $$|D^{\alpha}X_3(\eta)| \le C_0 d_0^m\, m!\, H,\ H < H_0$$

with C_0 and H independent of m and κ (and H_0 such that $\mathcal{T} \in \Omega_{e}^{R,\delta}$)

iii) The Jacobian determinant $|J|$ satisfies

(3.2d) $$C_1 H \kappa^2 \le |J| = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(\eta_1,\eta_2\ \eta_3)}\right| \le C_2 H \kappa^2$$

where C_1 and $\overline{C}_1$ are independent of κ and H.

Obviously $A > 1$ in (3.2a). Using (3.2a-d) we easily see that also

$$\frac{\kappa(\mathcal{T})}{\underline{\kappa}(\mathcal{T})} \geq A^* \tag{3.2e}$$

where $A^* > 1$ depending on constants in (3.2b,c,d). Let us note that the constants A, A^*, C_0, d_0, C_1, C_2, are not mutually independent; there are relations between them. Further we assume that these constants are the same for all elements of the meshes $\mathfrak{M} \in \mathcal{F}$.

2) <u>The element $\mathcal{T}_e$</u>. The element $\overline{\mathcal{T}}_e$ is the image of $\overline{D}$ by the mapping $M_{\mathcal{T}}$ satisfying the conditions (3.2b,c,d) and with one edge of $\mathcal{T}_e$ on the axis X_3 which is a part of the edge e of the domain Ω i.e. $\Gamma \cap \overline{\mathcal{T}}_e \neq 0$. For concreteness and without any loss of generality we assume that

$$X_1(-1,-1,\eta_3) = X_2(-1,-1,\eta_3) = 0$$

Remark 3.1: We see from (3.2a)(3.2e) that the size of the element $\mathcal{T}$ in (x_1,x_2) is of the order of the distance of $\mathcal{T}$ from the origin i.e. the edge e and the size of the element in the variable x_3 is of order H. This means that the element $\mathcal{T}$ is a "needle" i.e. has a very large aspect ratio. In the variables x_1,x_2 the element $\mathcal{T}$ has the same character as in the two dimensional case.

2) The elements in the vertex-edge neighborhood $\Omega^{R,\Phi}_{e,A}$.

1) <u>The element $\mathcal{T}$</u>. Let $\overline{\mathcal{T}} \subset \Omega^{R,\Phi}_{e,A}$ (i.e. $\overline{\mathcal{T}} \cap \Gamma = \emptyset$) be the image of $\overline{D}$ by an analytic mapping $M_{\mathcal{T}}$. We assume that

$$\mathcal{T} = \{x | x_i = X_i(\eta),\ \eta \in D,\ i = 1,2,3\}$$

where $X_i(\eta)$ are analytic functions with the properties spelled out below.

Let

$$\underline{\kappa}_1(\mathcal{T}) = \min_{x\in\overline{\mathcal{T}}} \rho(x), \tag{3.4a}$$

$$\kappa_1(\mathcal{T}) = \max_{x\in\overline{\mathcal{T}}} \rho(x), \tag{3.4b}$$

$$\underline{\kappa}_2(\mathcal{T}) = \min_{x\in\overline{\mathcal{T}}} \sin \varphi(x), \tag{3.4c}$$

$$\kappa_2(\mathcal{T}) = \max_{x\in\overline{\mathcal{T}}} \sin \varphi(x). \tag{3.4d}$$

About the element $\mathcal{T}$ and the mapping $M_{\mathcal{T}}$ we will assume the following:

i) There exist constants $1 < A_1, A_2 < \infty$ such that

$$\frac{\kappa_1(\mathcal{T})}{\underline{\kappa}_1(\mathcal{T})} \le A_1 \tag{3.5a}$$

$$\frac{\kappa_2(\mathcal{T})}{\underline{\kappa}_2(\mathcal{T})} \le A_2 \tag{3.5b}$$

ii) For $|\alpha| = m$, $m > 0$ an integer

$$|D^\alpha X_i(\eta)| \le C_0 d_0^m\, m!\, \kappa_1\kappa_2, \qquad i = 1,2 \tag{3.5c}$$

$$|D^\alpha X_3(\eta)| \le C_0 d_0^m\, m!\, \kappa_1 \tag{3.5d}$$

with C_0 independent of m, κ_1, κ_2.

iii) The Jacobian determinant $|J|$ satisfies

$$C_1 \kappa_1^3 \kappa_2^2 \le |J| = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(\eta_1,\eta_2,\eta_3)}\right| \le \overline{C}_1 \kappa_1^3 \kappa_2^2 \tag{3.5e}$$

where C_1 and $\overline{C}_1$ are independent of κ_1, κ_2.

Combining (3.5a–e) there also exist A_1^*, A_2^* so that

$$\frac{\kappa_1(\mathcal{T})}{\underline{\kappa}_1(\mathcal{T})} \ge A_1^* \tag{3.5f}$$

$$\frac{\kappa_2(\mathcal{T})}{\underline{\kappa}_2(\mathcal{T})} \ge A_2^*, \tag{3.5g}$$

where A_1^*, $A_2^* > 1$ depend on the constants in (3.5a-e).

Remark 3.2. We see from (3.5a) (3.5b) that the element proportions depend on κ_1 and κ_2 and its size is of the order of the distance to the origin and axis x_3.

<u>The element</u> $\mathcal{T}_{e,A}$. We assume that $\mathcal{T}_{e,A}$ is image of $\bar{D}$ by the mapping $M_{\mathcal{T}}$ satisfying the properties (3.5c-e) but instead (3.5ab) we will assume only (3.5a) and that one edge of $\mathcal{T}_{e,A}$ lies on the axis x_3 which coincides with the edge e i.e. $\bar{\mathcal{T}}_{e,A} \cap \Gamma \neq \phi$ but $A \notin \bar{\mathcal{T}}_{eA}$. For concreteness we will assume that $X_1(-1,-1,\eta_3) = X_2(-1,-1,\eta_3) = 0$.

Remark 3.3. We described only the elements for which $A \notin \bar{\mathcal{T}}$.

Elements $\mathcal{T}$ for which $A \in \bar{\mathcal{T}}$ will be addressed in the subsection 5 below.

3) The elements in the region Ω_A^R.

Let $\bar{\mathcal{T}} \subset \Omega_A^R$ be as before the image of $\bar{D}$ so that

$$\mathcal{T} = \{x \mid x_i = X_i(\eta),\ \eta \in D,\ i = 1,2,3\}$$

and $X_i(\eta)$ are analytic functions with the following properties. Let

$$\underline{\kappa}(\mathcal{T}) = \min_{x\in\bar{\mathcal{T}}} \rho(x), \tag{3.6a}$$

$$\kappa(\mathcal{T}) = \max_{x\in\bar{\mathcal{T}}} \rho(x). \tag{3.6b}$$

Then we will assume the following:

i) There exists a constant $1 < A < \infty$ such that

$$\frac{\kappa(\mathcal{T})}{\underline{\kappa}(\mathcal{T})} \le A. \tag{3.7a}$$

ii) For $|\alpha| = m$, $m > 0$ integer

$$|D^\alpha X_i| \le C_0 d_0^m\, m!\, \kappa,\ i = 1,2,3 \tag{3.7b}$$

with C_0 independent of m and κ.

iii) The Jacobian determinant $|J|$ satisfies

(3.7c) $$C\kappa^3 \le |J| = \left|\frac{\partial(X_1,X_2,X_3)}{\partial(\eta_1,\eta_2,\eta_3)}\right| \le \bar{C}\kappa^3$$

where C and $\bar{C}$ are independent of κ. Combining (3.7abc) we also get

(3.7d) $$\frac{\kappa(\mathcal{T})}{\underline{\kappa}(\mathcal{T})} \ge A^*$$

where $A^* > 1$ depending on the constants in (3.7abc).

Remark 3.4. We see that the element has not large aspect ratio and its size depend on the distance from the vertex A.

Remark 3.5. Here we do not address the element $\mathcal{T}$ with $A \in \bar{\mathcal{T}}$; it will be addressed in the subsection 5 below.

4) The elements in the regular region Ω_0^R.

Once more let $\mathcal{T} \in \Omega_0^R$ and $\mathcal{T}$ be image of D by the mapping $M_{\mathcal{T}} = \{X_i\}$ $i = 1,2,3$. Let

(3.8) $$\kappa = \max_{x\in\mathcal{T}} (\text{dist } x,\Gamma).$$

i) We will assume that for $|\alpha| = m$, $m > 0$ integer

(3.9a) $$|D^\alpha X_i(\eta)| \le C_0 d_0^m\, m!\, \kappa$$

with C_0 independent of m and κ.

ii) The Jacobian determinant $|J|$ satisfies

(3.9b) $$C\kappa^3 \le |J| = \left|\frac{\partial(X_1,X_2,X_3)}{\partial(\eta_1,\eta_2,\eta_3)}\right| \le \bar{C}\kappa^3$$

where C and $\bar{C}$ are independent of κ.

From (3.9a,b) it follows that the volume of any $\mathcal{T} \in \Omega_0^R$ is bounded from below and so only finite number of elements $\mathcal{T} \in \Omega_0^R$ exist and κ is equivalent to the diameter of $\mathcal{T}$.

Remark 3.6. All elements in Ω_0^R have no large aspect ratio.

5) The vertex element $\mathcal{T}_A$.

So far we did not address the element $\mathcal{T}_A$ such $A \in \bar{\mathcal{T}}_A$. In all previous cases we have assumed that the (entire) element $\mathcal{T}$ is in one (or more) domains. About the vertex element we will assume that in general $\mathcal{T}$ is not necessarily in the interior of any of the domains $\Omega_{\mathcal{e},A}^{R,\Phi}$, Ω_A^R. Let $A \in \bar{\mathcal{T}}$. We will assume that $\mathcal{T}_A$ is the image of D by the mapping $M_{\mathcal{T}} = \{X_i\}$ $i = 1,2,3$. Denoting $\operatorname{diam} \mathcal{T} = \kappa$ then we will assume that (3.5 bcde) (3.7 ab) hold.

3.4 The mesh and the finite elements

We will consider a family $\mathcal{F}$ of meshes on Ω. A mesh $\mathfrak{M} \in \mathcal{F}$ will be a partition of Ω into the set of elements $\mathcal{T} \in \mathcal{M}$, $\bigcup_{\mathcal{T}\in\mathfrak{M}} = \bar{\Omega}$. We will assume that the elements $\mathcal{T}$ are curvilinear bricks which properties were described in the Section 3.3. For simplicity we are restricting ourselves to the case of brick elements only, although the theory holds for tetrahedrons in the similar way as in two dimensions the triangular elements were treated.

To every $\mathcal{T} \in \mathfrak{M}$ we associate an analytic mapping $M_{\mathcal{T}}$ which is defined on the standard cube and $\mathcal{T}$ is the image of the cube by the mapping $M_{\mathcal{T}}$. In an obvious way we can speak about the vertices, edges or faces of the elements $\mathcal{T} \in \mathfrak{M}$. As usually we will assume that if $\mathcal{T}_i, \mathcal{T}_j \in \mathfrak{M}$ then either $\overline{\mathcal{T}}_i \cap \overline{\mathcal{T}}_j = \phi$ or $\overline{\mathcal{T}}_i \cap \overline{\mathcal{T}}_j$ is a common vertex of $\mathcal{T}_i$ and $\mathcal{T}_j$ or $\overline{\mathcal{T}}_i \cap \overline{\mathcal{T}}_j$ is a (common) entire edge or entire face of $\mathcal{T}_i$ and $\mathcal{T}_j$. If $\mathcal{T}_i, \mathcal{T}_j$ have common edge or face we assume that the mapping $M_{\mathcal{T}_i}$ and $M_{\mathcal{T}_j}$ have the usual properties guaranteeing the continuity of the finite elements. The meshes consisting of elements described above have a special character of a geometrical mesh in the neighborhood of edges and vertices. This follows from (3.2a) (3.2e), (3.5ab), (3.5fg) (3.7a) (3.7d).

We will consider the family $\mathcal{F}$ of meshes characterized by two parameters (σ, n) analogously as in two dimensions. Consider the edge $e = \{x_1 = x_2 = 0, 0 < x_3 < 1\}$ and the vertex $A = (0,0,0)$.

1) The mesh in the edge region $\Omega_{e}^{R,\delta}$. Here the mesh is geometric in (x_1, x_2) with

$$\kappa(\mathcal{T}_e) \le \bar{C}\sigma^n.$$

In the x_3 variable the series of the elements are independent of n, i.e., the mesh is fixed in the x_3 direction; see (3.1), (3.2). We can

(analogously as in two dimensional case) speak about the layer (in x_1,x_2) and the level which is a sequential number of element in x_3 direction. The mesh hence has n-layers and α_0(independent of n) levels. Any layer has at most K elements and hence the total number of elements in $\Omega_{e}^{R,\delta}$ is $O(n)$ similarly as in 2 dimensions.

2) The mesh in the vertex-edge region $\Omega_{e,A}^{R,\Phi}$.

The mesh has geometric character in (x_1,x_2) and also in x_3. We can also speak here about the layers (in x_1,x_2) and levels in x_3. The mesh (see (3.4), (3.5)) is such that

$$\kappa_2(\mathcal{T}_{e,A}) \le \bar{C}\sigma^n$$

and

$$C\sigma^n \le \min_{\mathcal{T}\subset\Phi_{e,A}^{R,\Phi}} \kappa_1(\mathcal{T}) \le \bar{C}\sigma^n.$$

By the analogous argument as in the two dimenional settings we can see that the number of elements in $\Omega_{e,A}^{R,\Phi}$ is of order $O(n^2)$.

3) The mesh in the vertex region Ω_{A}^{R}.

Here the mesh is geometric in the $\rho(x)$ variable and quasiuniform in the two other variables (see (3.6), (3.7)) with

$$C\sigma^n \le \min_{\mathcal{T}\subset\Phi_{A}^{R}} \kappa(\mathcal{T}) \le \bar{C}\sigma^n.$$

Hence the number of elements in $\Omega_{e,A}^{R,\Phi}$ is of order $O(n)$.

4) The mesh in the regular region Ω_{0}^{R}.

Here the mesh has number of elements independent of n. Hence the number of elements in Ω_{0}^{R} is of order $O(1)$.

5) The set of vertex elements.

Here we have

$$C\sigma^n \le \text{diam }(\mathcal{T}_0) \le \bar{C}\sigma^n.$$

Obviously we have $O(1)$ vertex elements.

Remark 4.1. Similarly as in 2 dimensions (see remarks 4.1 and 4.2 in the Section 2) we have analogous inequalities for $\kappa(\mathcal{T})$. For example let the element $\mathcal{T} \subset \Omega^{R,\Phi}_{\mathcal{e},A}$ be located in the k^{th} layer and j^{th} level. Then

$$\underline{C}\sigma^n\rho_1^{-k} \le \kappa_2(\mathcal{T}) \le \bar{C}\sigma^n\rho_2^{-k},$$

and

$$C\sigma^n\rho_1^{-k} \le \kappa_1(\mathcal{T}) \le \bar{C}0^n\rho_2^{-k}$$

with $0 < \rho_1, \rho_2 < 1$.

In practice we construct the mesh with $\rho_1 = \rho_2 = \sigma$. The number of elements in the region $\Omega^{R,\Phi}_{\mathcal{e},A}$ can be estimated analogously as in the two dimensional case. Similar estimates for the elements located in other regions are analogous.

The shape functions are given on the master elements which in our case is the unit cube D. We will assume that the elements are of degree $p(\mathcal{T})$ in every variable η_i, $i = 1,2,3$ separately. For simplicity we will assume that $p(\mathcal{T}) = p$ i.e. the degrees are uniform although we could consider $p(\mathcal{T})$ different for every element and different in every direction. This would increase the effectiveness of the method. The space of the finite elements is denoted by $S(\mathfrak{M},p) = S(\sigma,n,p) \subset H^1_0(\Omega)$.

3.5. The rate of convergence of the h-p version.

In the Section 3.4 we have described the family of the meshes $\mathcal{F}$ characterized by two parameters (σ,n). Given the exact solution u_0 with the properties given in the section 3.2 we follow the framework outlined in the Section 2.7. First we construct the approximation element by element

and then the corrections which guarantee the conformity of the elements.

By a detailed analysis of the approximation of the functions $u \in \mathcal{B}(\Omega)$ we can show (see [16]) that for

$$p(\mathcal{T}) = p = \mu n, \ \mu > 0$$

there exists a function $\chi \in S(\sigma,n,p)$ so that in any regions $\Omega_{e}^{R,\delta}$, $\Omega_{e,A}^{R,\Phi}$, Ω_{A}^{R}, Ω_{0}^{R} we have

$$(5.1) \qquad |u_0 - \chi|_{H^1(\Omega)} \le C\,\sigma^{n\gamma}$$

where $\gamma > 0$ depends on the constants β_i in $\mathcal{B}_{\beta}(\Omega)$ introduced in Section 3.2.

Let us now estimate the error in the term of degrees of freedom.

1) The region $\Omega_{e}^{R,\delta}$.

As we have seen above there are $0(n)$ elements in the region $\Omega_{e}^{R,\delta}$. Every element has $0(p^3)$ degrees of freedom $N(\mathcal{T})$. Denoting by $N(\Omega_{e}^{R,\delta})$ the number of degrees of freedom in $\Omega_{e}^{R,\delta}$ we get

$$N(\Omega_{e}^{R,\delta}) = 0(np^3) = 0(n^4)$$

when we used $p(\mathcal{T}) = \mu n$. Hence we have

$$(5.1) \qquad |u_0-\chi|_{H^1(\Omega_{e}^{R,\delta})} \le C\sigma^{\bar{\gamma}N^{1/4}} = Ce^{-\gamma^* N^{1/4}}.$$

2) The region $\Omega_{e,A}^{R,\Phi}$.

As stated above the total number of elements in $\Omega_{e,A}^{R,\Phi}$ is $0(n^2)$. Hence $N(\Omega_{e,A}^{R,\Phi}) = 0(n^2p^3) = 0(n^5)$ and

$$(5.2) \qquad |u_0-\chi|_{H^1(\Omega_{e,A}^{R,\Phi})} \le C\sigma^{\bar{\gamma}N^{1/5}} = Ce^{-\gamma^* N^{1/5}}.$$

3) The region Ω_{A}^{R}.

In Ω_{A}^{R} there are $0(n)$ elements and hence

$$(5.3) \qquad |u_0-\chi|_{H^1(\Omega_A^R)} \le C\sigma^{\bar{\gamma}N^{1/4}} - Ce^{-\gamma^*N^{1/4}}.$$

4) The region Ω_0^R.

The number of elements in this region is $O(1)$ and hence

$$(5.4) \qquad |u_0-\chi|_{H^1(\Omega_0^R)} \le C\sigma^{\bar{\gamma}N^{1/3}} = Ce^{-\gamma^*N^{1/3}}.$$

5) The vertex elements.

Here we have only $O(1)$ elements and hence the error is of order $Ce^{-\gamma^*N^{1/3}}$.

Hence we have proven

<u>Theorem</u> 3.5.1. Let $u_0 \in \mathcal{B}_\beta(\Omega)$ (see Section 3.2) be the exact solution of the problem (2.7). Further let the meshes $\mathfrak{M}(\sigma,n)$, $\sigma < 1$ are as in the Section 3.4 and let $p(\mathcal{T}) = \mu n$, $\mu > 0$, properly chosen. Then

$$(5.5) \qquad \|u_0 - u_{FE}\|_{E(\Omega)} \le C \inf_{\chi\in S(\sigma,\eta,p)} |u - \chi|_{H^1(\Omega)} \le Ce^{-\gamma N^{1/5}}$$

where $N = \dim S(\sigma,\eta,p)$ is the number of degrees of freedom, γ and C depend on $\mathcal{B}_\beta(\Omega)$, the distortion of the elements the solution u_0 and the domain Ω but are independent of N.

Remark 5.1. Theorem 3.5.1 follows from (5.2)(5.3)(5.4). We see that the factor $N^{1/5}$ is due to the vertex-edge singularly of the solution. Hence we can expect that the rate $e^{-\gamma N^{1/5}}$ will be visible only for large N i.e. high accuracy and for smaller accuracies we can expect the rate $e^{-\gamma N^{\alpha}}$, $\frac{1}{4} < \alpha < \frac{1}{5}$. We will see it in the next section.

Remark 5.2. We assumed in the Theorem 5.1 that the (pull-back) polynomials are at the same degree p in η_1,η_2,η_3 in all elements. It is possible to

prove that the error in (5.5) holds with better constants if the degree of the elements is different in different elements and in different directions in η_i, $i = 1,2,3$.

Remark 5.3. In [12] we have proven that in the one dimension and the function of the analogous type as here the exponential rate is $e^{-\gamma N^{\alpha}}$, $\alpha = 1/2$ and α cannot be made larger for any mesh and any degrees of the polynomials. We conjecture that the exponential rate $e^{-\gamma N^{\alpha}}$, $\alpha = \frac{1}{5}$ in 3 dimensions cannot be improved also if any mesh and any anisotropic degree distributions would be considered.

Similarly we can prove

Theorem 3.5.2. Let u_0 be the exact solution of the Problem 2.7 and be analytic on $\bar{\Omega}$. Consider the meshes $\mathfrak{M}(\sigma,\eta)$ with $\sigma = 1$ (hence $\mathfrak{M}(\sigma,\eta)$ has only finite number of elements independently of n) and let $p(\mathcal{T}) = \mu n$, $\mu > 0$. Then

$$\|u_0 - u_{FE}\|_E \le C \inf_{X \in S(\sigma,\eta,p)} |u - \chi| \le C e^{-\gamma N^{1/3}}.$$

Remark 5.4. For the results mentioned above see also [17].

3.6. A numerical example

In this section we will show a typical numerical example. Consider domain shown in the Fig. 3.6.1 and 3.6.2 and the problem

$$\Delta u = 0 \quad \text{on} \quad \Omega.$$

The boundary conditions are

i) $u = 0$ on BFHGQKD, ABDC, AEFB, ACNE, PRKQ, SGHF, BFHGQKD,

ii) $\frac{\partial u}{\partial n} = 0$ on SLRPRQG, MJHFEN, MNCDKRL,

iii) $\frac{\partial u}{\partial n} = \cos\left(\frac{\pi x_1}{4}\right)$ on the bottom surface.

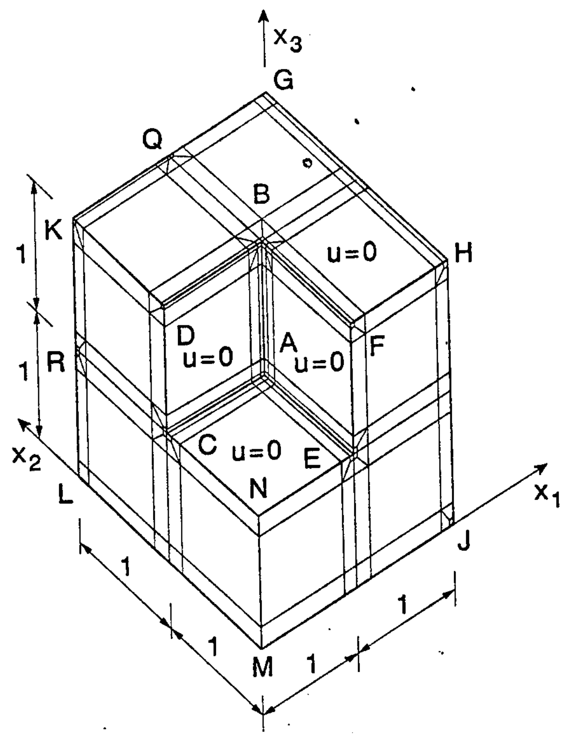

Fig. 3.6.1 The domain Ω and the scheme of the mesh

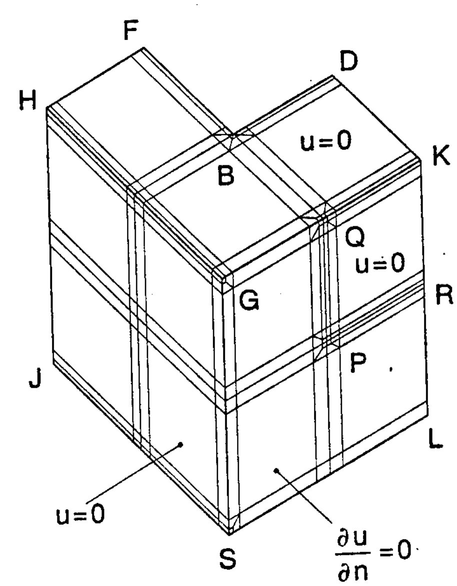

Fig. 3.6.2 The domain Ω and the scheme of the mesh

We constructed the geometrical meshes of the type described in the previous subsections with $\sigma = 0.15$. One of the scheme of the meshes (elements

surfaces) is shown in the figures. Analogously as in the two dimensional case for every mesh characterized by $n = i$, $i = 1,\dots,7$ we solve the problem by $p = 2,\dots,8$. Theorem 3.5.1 suggests that the error measured in the energy norms behaves as $e^{-\gamma N^{1/\alpha}}$ where $\alpha \le 5$ and we can expect that we will see $4 < \alpha \le 5$ in the computation. Hence we plot the relative error $\|e\|_{ER}$ in the scale $\log \|e\|_{ER} \times N^{1/\alpha}$. Then we will expect that of the error of the h-p version will decay linearly. In the Fig. 3.7.3 we show the errors for $\alpha = 4$ and in Fig. 3.7.4 for $\alpha = 5$.

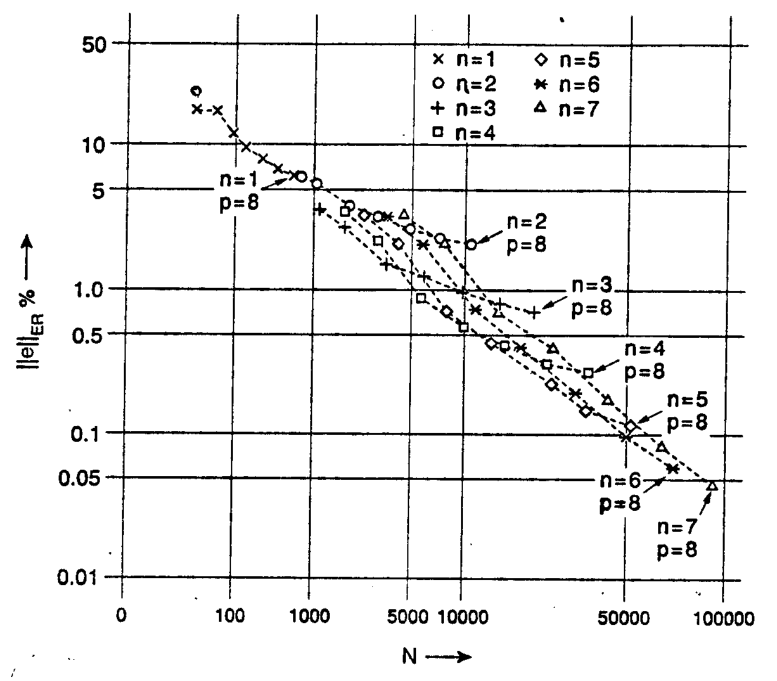

Fig. 3.6.3 The error as function of the degrees of freedom in the scale $\log \|e\|_{ER} \times N^{1/4}$

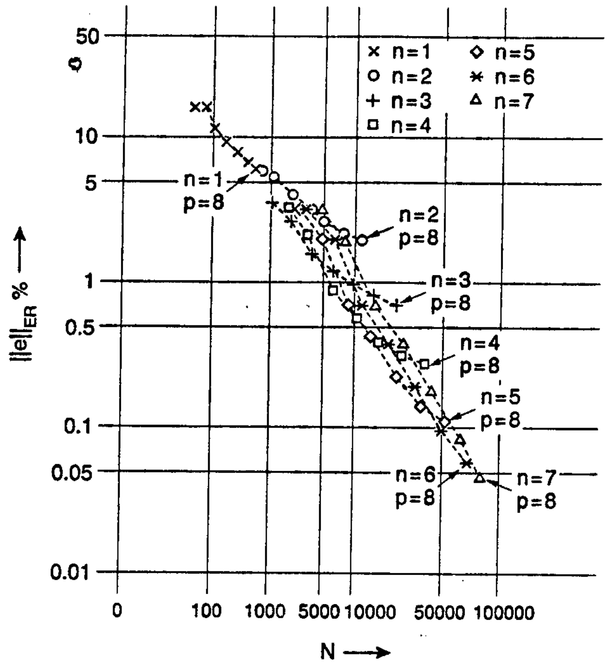

Fig. 3.6.3 The error as function of the degrees of freedom in the scale $\log \|e\|_{ER} \times N^{1/5}$

We see that the h-p version converges exponentially with $\|e\|_{ER} \approx Ce^{-\gamma N^{1/\alpha}}$ $4 < \alpha \leq 5$ as expected in the range of the computation.

Remark 6.1. The exact solution is of course not known. We computed the strain energy $\mathcal{E}$ of the exact solution by extrapolation and the error of the finite solution is then $\|e\|_E^2 = \mathcal{E}(u_{EXACT}) - \mathcal{E}(u_{FE})$.

Remark 6.2. The above example was computed by Dr. B. Anderson (Aeronautical Institute of Sweden) using the program STRIPE (developed in the Aeronautical Institute).

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Further information may be obtained from **Professor I. Babuška**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431.